HW 1.

2.1

(a) $g(t) = A\cos(2\pi f_c t)\,\text{rect}\left(\frac{t}{T}\right)$

$f_c = \frac{1}{T_s} = \frac{1}{2T}$

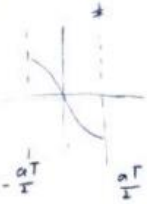
$\Rightarrow G(f) = A \cdot \mathcal{F}\{\cos(2\pi f_c t)\} \circledast \mathcal{F}\{\text{rect}(\frac{t}{T})\}$

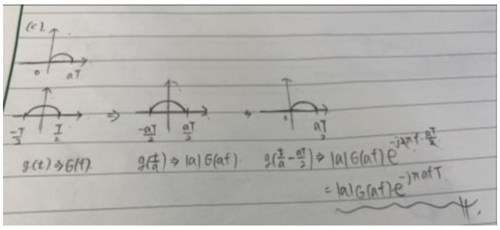
$= A \cdot \{\frac{1}{2}\delta(f-f_c) + \frac{1}{2}\delta(f+f_c)\} \cdot T\,\text{sinc}(Tf)$

$= \frac{AT}{2}\{\text{sinc}(T(f+f_c)) + \text{sinc}(T(f-f_c))\}$ *

(b) $g_2(t) = g(t - \frac{T}{2})$

$= G(f)e^{-j2\pi f\frac{T}{2}} = G(f)e^{-j\pi fT}$ *





(d) $g_4(t) = -g(t + \frac{T}{2})$

$\Rightarrow -G(f)e^{j2\pi f\frac{T}{2}}$

$= -G(f)e^{j\pi fT}$ *

(e) $g_5(t) = g_2(t) + g_4(t)$

$G(f) = G(f)e^{-j\pi fT} - G(f)e^{j\pi fT}$

$= G(f)\,(\cos(\pi fT) - j\sin(\pi fT) - (\cos(\pi fT) + j\sin(\pi fT)))$

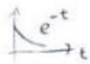
$= G(f)\,(-2j\sin(\pi fT))$ *

2.2

$g(t) = e^{-t}\sin(2\pi f_c t)\,u(t)$

$= e^{-t}u(t) \cdot \sin(2\pi f_c t)$

$\mathcal{F} \Rightarrow \mathcal{F}\{e^{-t}u(t)\} \circledast \{\frac{1}{2j}\delta(f-f_c) - \frac{1}{2j}\delta(f+f_c)\}$



$= \int_0^\infty e^{-t}e^{-j2\pi ft}\,dt$

$= \int_0^\infty e^{-t(1+j2\pi f)}\,dt$

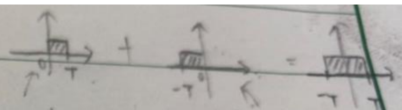
$= \frac{1}{-1-j2\pi f}[0-1]$

$= \frac{1}{1+j2\pi f}$

$= \frac{1}{2j}\left[\frac{1}{1+j2\pi(f-f_c)} - \frac{1}{1+j2\pi(f+f_c)}\right]$ *

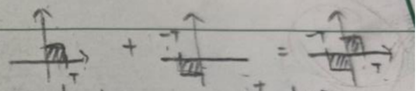
3.

$g(t) = A \cdot rect(\frac{t}{T} - \frac{1}{2})$



(a) $g_e(t) = \frac{1}{2}[g(t) + g(-t)] = \frac{1}{2}[A\, rect(\frac{t}{T} - \frac{1}{2}) + A\, rect(\frac{-t}{T} - \frac{1}{2})]$

$= \frac{A}{2} rect(\frac{t}{2T})$



$g_o(t) = \frac{1}{2}[g(t) - g(-t)] = \frac{1}{2}[A\, rect(\frac{t}{T} - \frac{1}{2}) - A\, rect(\frac{-t}{T} - \frac{1}{2})]$

$-T \sim 0$. $\quad 0 \sim -T$

$= \frac{A}{2}[rect(\frac{t}{T} - \frac{1}{2}) - rect(\frac{t}{T} + \frac{1}{2})]$.

(b) 由图.

$G_e(f) = \mathcal{F}\{\frac{A}{2} rect(\frac{t}{2T})\} = \frac{A}{2} \cdot 2T\, sinc(2Tf)$

$= AT\, sinc(2fT)$ #

$G_o(f) = \mathcal{F}\{\frac{A}{2}[rect(\frac{t - \frac{T}{2}}{T}) + rect(\frac{-t - \frac{1}{2}T}{T})]$

$= \frac{AT}{2}[sinc(Tf) e^{-j2\pi \frac{T}{2} f} + sinc(Tf) e^{-j2\pi \frac{T}{2} f}]$

$\cos - j \quad + (\cos - j)$

$= \frac{AT}{2} sinc(Tf)[-2j \sin(\pi Tf)]$

$= \frac{AT}{j} sinc(fT) \sin(\pi fT)$ #

2-4. $G(f) = \begin{cases} 1 \cdot e^{j(-\frac{\pi}{2})} = -j, & 0 < f < W \\ 1 \cdot e^{j(\frac{\pi}{2})} = j, & -W < f < 0 \end{cases}$

$g(t) = \int_0^W -j e^{j2\pi ft} df + \int_{-W}^0 j e^{j2\pi ft} df$

$= -j \frac{1}{j2\pi t}(e^{j2\pi ft}|_{f=0}^{W}) + j \frac{1}{j2\pi t}(e^{j2\pi ft}|_{-W}^{0})$

$= \frac{-1}{2\pi t}(e^{j2\pi tW} - 1) + \frac{1}{2\pi t}(1 - e^{-j2\pi tW})$

$= \frac{-1}{2\pi t} e^{j2\pi tW} + \frac{1}{2\pi t} + \frac{1}{2\pi t} - \frac{1}{2\pi t} e^{-j2\pi tW}$

$\cos + j\sin \quad \cos - j\sin$

$= -\frac{1}{2\pi t}[2\cos(2\pi tW)] + \frac{1}{\pi t} \quad \sin^2 x = \frac{1 - \cos 2x}{2}$

$= \frac{1 - \cos(2\pi tW)}{\pi t}$ # $\quad = \frac{2\sin^2(\pi tW)}{\pi t} = 2W\, sinc(tW) \sin(\pi tW)$ #

#2.5

(a) even 係 $\Rightarrow g(t) = g(-t)$. $\int_{-\infty}^{\infty} g(-t) e^{-j2\pi f t} dt$

$\Rightarrow \Downarrow F.T.$ $G(f) = G^*(f)$ $\quad$ $-t = \tau$ $\quad \int_{-\infty}^{\infty} g(\tau) e^{j2\pi f \tau} d\tau$

$a+jb \quad a-jb \quad = G^*(f)$

$\Rightarrow G(f)$ is real #

odd $\Rightarrow g(-t) = -g(t)$

$\Rightarrow \Downarrow F.T.$ $G^*(f) = -G(f)$

$a-jb = -(a+jb) = -a-jb$

$\Rightarrow G(f)$ is image #

(b) $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

$\frac{d^n}{df^n} X(f) = \int_{-\infty}^{\infty} x(t) \frac{d^n}{df^n} e^{-j2\pi f t} dt$

$= \int_{-\infty}^{\infty} x(t) (-j2\pi t)^n e^{-j2\pi f t} dt$

$= (-j2\pi)^n \int_{-\infty}^{\infty} \underbrace{x(t) t^n} e^{-j2\pi f t} dt$

$= (-j2\pi)^n \mathcal{F}\{x(t) \cdot t^n\}$

$\Rightarrow \mathcal{F}\{x(t) \cdot t^n\} = \left(\frac{j}{2\pi}\right)^n \frac{d^n X(f)}{df^n}$

(c) $\Rightarrow \int_{-\infty}^{\infty} t^n x(t) e^{-j2\pi f t} dt$

由題 $f=0$ $\quad = \left(\frac{j}{2\pi}\right)^n \frac{d^n X(0)}{df^n}$ #

$g(t) \to G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt$

$g^*(t) \to G(f) = \int_{-\infty}^{\infty} g^*(t) e^{-j2\pi f t} dt$

$= \left(\int_{-\infty}^{\infty} g(t) e^{j2\pi f t} dt\right)^*$

$G^*(-f)$

(d)

$\int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt$

$\int_{-\infty}^{\infty} g_1(t) \int_{-\infty}^{\infty} G_2^*(-f) e^{j2\pi f t} df\, dt$

$= \int_{-\infty}^{\infty} g_1(t) \int_{-\infty}^{\infty} G_2^*(f') e^{-j2\pi f' t} df'\, dt$

$= \int_{-\infty}^{\infty} G_2^*(f) G_1(f) df$ #

(e) $g_1(t) g_2^*(t)$

$\xrightarrow{F.T.} \mathcal{F}\{g_1(t)\} * \mathcal{F}\{g_2^*(t)\}$

$= G_1(f) * G_2^*(-f)$

$= \int_{-\infty}^{\infty} G_1(\lambda) G_2^*(-(f-\lambda)) d\lambda$

$= \int_{-\infty}^{\infty} G_1(\lambda) G_2^*(\lambda - f) d\lambda$ #

#2.6

(a) F.T. $g_1(t) * g_2(t) \xrightarrow{F.T.} G_1(f) \cdot G_2(f)$

$= G_2(f) \cdot G_1(f) \xrightarrow{F.T.^{-1}} g_2(t) * g_1(t)$ #

(b) F.T.

$g_1(t) * [g_2(t) * g_3(t)] \xrightarrow{F.T.} G_1(f) \cdot (G_2(f) \cdot G_3(f))$

$= [G_1(f) \cdot G_2(f)] G_3(f) \xrightarrow{F.T.^{-1}} (g_1(t) * g_2(t)) * g_3(t)$

(c) F.T.

$g_1(t) * [g_2(t) + g_3(t)] \xrightarrow{F.T.} G_1(f) \cdot [G_2(f) + G_3(f)]$

$= G_1(f) G_2(f) + G_1(f) G_3(f) \xrightarrow{F.T.^{-1}} g_1(t) * g_2(t) + g_1(t) * g_3(t)$ #

Prom 7 by $F\{\frac{d^n x(t)}{dt^n}\} = (j2\pi f)^n X(f)$

pf, $\frac{d}{dt}[g_1(t) * g_2(t)] \xrightarrow{F.T.} (j2\pi f)\ F\{g_1(t) * g_2(t)\}$

$= (j2\pi f)\,[G_1(f) \cdot G_2(f)]$

$= [j2\pi f \cdot G_1(f)] \cdot G_2(f)$

$\downarrow F^{-1}.T$

$= \left(\frac{d}{dt} g_1(t)\right) * g_2(t)$ \#

(b), by $F\{\int_{-\infty}^{t} x(\tau)\,d\tau\} = \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\,\delta(f)$

$\Rightarrow \int_{-\infty}^{t} [g_1(\tau) * g_2(\tau)]\,d\tau \xrightarrow{F.T} \frac{1}{j2\pi f}\{G_1(f)\, G_2(f)\}$

$= \{\frac{1}{j2\pi f} G_1(f)\} \cdot G_2(f)$

$\xrightarrow{F^{-1}.T} \left(\int_{-\infty}^{t} g_1(\tau)\,d\tau\right) * g_2(t)$ \#

Prom 8

(1) $g(t) = rect(\frac{t}{T})$

$G(f) = T\,sinc(Tf)$

$\Rightarrow g_{new}(t) = \frac{1}{T} rect(\frac{t}{T})$

$G_{new}(f) = sinc(Tf)$ → 單位化

$\Rightarrow \lim_{T\to 0} g_{new}(t) \to \delta(t)$

$\lim_{T\to 0} G(f) \to 1$

(2) $g(t) = 2W\,sinc(2Wt)$

$G(f) = rect(\frac{f}{2W})$

$\lim_{W\to\infty} g(t) = \delta(t)$

$\lim_{W\to\infty} G(f) = 1$

Prom 9

$G(f) = \frac{1}{2} sgn(f) + \frac{1}{2}$

$\to g(t) = \frac{1}{2}\delta(t) + \frac{j}{2\pi t}$ \#

by $sgn(t) = \frac{1}{j\pi f} \Rightarrow sgn(f) = -\frac{1}{j\pi t} = \frac{j}{\pi t}$

10. Given: $\int_{-\infty}^{\infty} |x(t)|^2 < \infty$ and $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

by Rayleigh's Thm $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df < \infty$

$$\left|\int_{-\pi}^{\pi} f(x)\, g^*(x)dx - \int_{-\pi}^{\pi} S_N(f)\, g^*(x)dx\right| = \left|\int_{-\pi}^{\pi} [f(x) - S_N(f)]\, g^*(x)dx\right|$$
$$\leq \int_{-\pi}^{\pi} |[f(x) - S_N(f)]\, g^*(x)| dx$$
by Schwarz inequality
$$\leq \left(\int_{-\pi}^{\pi} |f(x) - S_N(f)|^2 dx \cdot \int_{-\pi}^{\pi} |g^*(x)|^2 dx\right)^{\frac{1}{2}} \to 0$$

$\Rightarrow$ $y(t) = x(t) \circledast h(t)$

$Y(f) = H(f)\, X(f)$

by Rayleigh's Thm

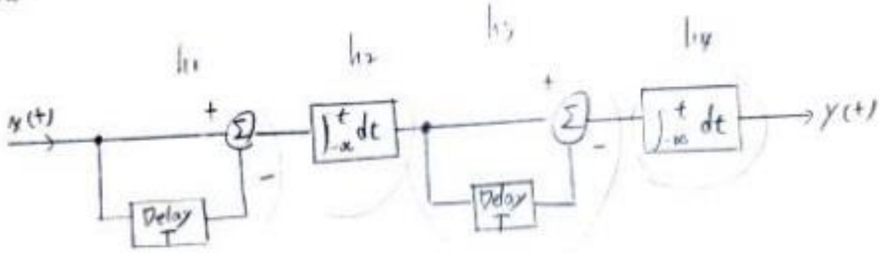
$\Rightarrow$ $\int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |Y(f)|^2 df = \int_{-\infty}^{\infty} Y(f)\, Y^*(f)\, df$

$= \int_{-\infty}^{\infty} H(f) X(f) \cdot H^*(f) X^*(f)\, df$

$= \int_{-\infty}^{\infty} |X(f)|^2 |H(f)|^2 df \underset{柯西不等式}{\leq} \underbrace{\int_{-\infty}^{\infty} |X(f)|^2 df}_{<\infty} \cdot \underbrace{\int_{-\infty}^{\infty} |H(f)|^2 df}_{<\infty}$

$\therefore \int_{-\infty}^{\infty} |y(t)|^2 dt < \infty$ \#

11.



$y(t) = x(t) \circledast h_1(t) \circledast h_2(t) \circledast h_3(t) \circledast h_4(t)$

$\Rightarrow Y(f) = X(f) \cdot H_1(f) \cdot H_2(f) \cdot H_3(f) \cdot H_4(f)$

by ① $h_1(t) = h_3(t) = \delta(t) - \delta(t-T)$

$\rightarrow H_1(f) = H_3(f) = 1 - e^{-j2\pi f T}$

② $h_2(t) = h_4(t) = \int_{-\infty}^{t} dt$

$\rightarrow H_2(f) = H_4(f) = \frac{1}{j2\pi f}$

$\therefore H(f) = \frac{Y(f)}{X(f)} = H_1(f) \cdot H_2(f) H_3(f) H_4(f)$

$= (1 - e^{-j2\pi f T})^2 \cdot \left(\frac{1}{j2\pi f}\right)^2$

$= \frac{-1}{(2\pi f)^2} \left[1 - 2e^{-j2\pi f T} + e^{-j4\pi f T}\right]$ \#