Signal Design for Band-Limited Channels

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Introduction

- We consider the problem of signal design when the channel is band-limited to some specified bandwidth of W Hz.
- The channel may be modeled as a linear filter having an equivalent low-pass frequency response $C(f)$ that is zero for $|f| > W$.
- Our purpose is to design a signal pulse $g(t)$ in a linearly modulated signal, represented as

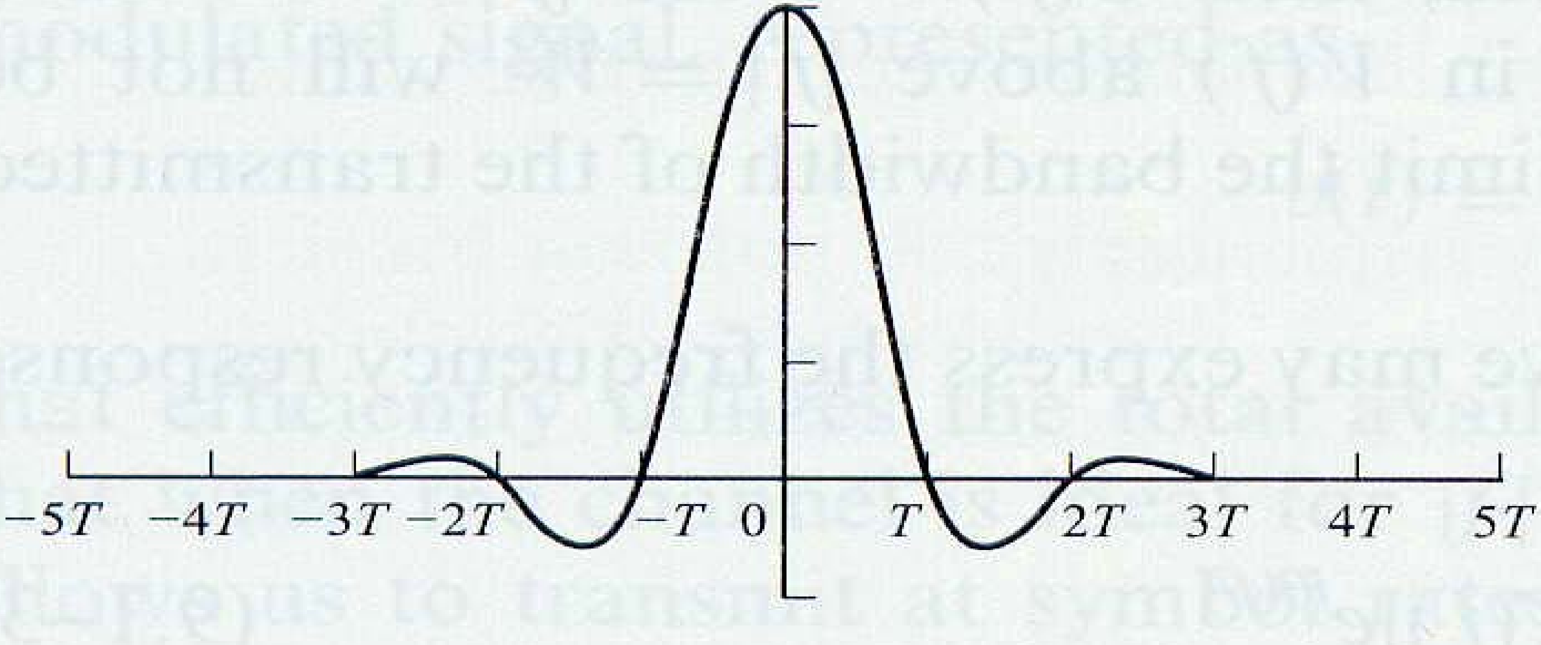
$$v(t) = \sum_n I_n g(t - nT)$$

that efficiently utilizes the total available channel bandwidth W.

  - When the channel is ideal for $|f| \leq W$, a signal pulse can be designed that allows us to transmit at symbol rates comparable to or exceeding the channel bandwidth W.
  - When the channel is not ideal, signal transmission at a symbol rate equal to or exceeding W results in *inter-symbol interference* (ISI) among a number of adjacent symbols.

Characterization of Band-Limited Channels

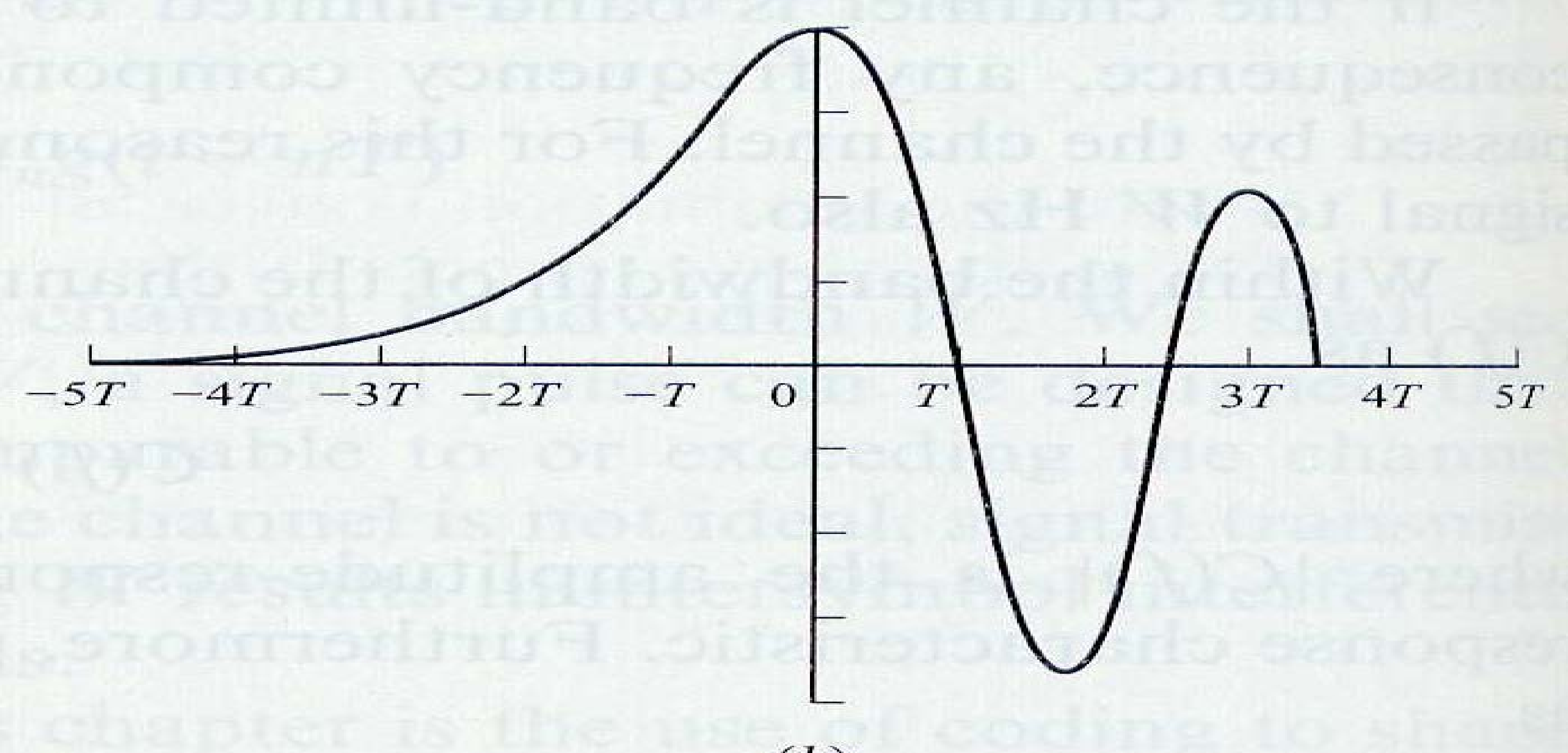
- Fig. (a) is a band-limited pulse having zeros periodically spaced in time at ±*T*, ±2*T*, etc.
- If information is conveyed by the pulse amplitude, as in PAM, for example, then one can transmit a sequence of pulses, each of which has a peak at the periodic zeros of the other pulses.



(*a*)

Characterization of Band-Limited Channels

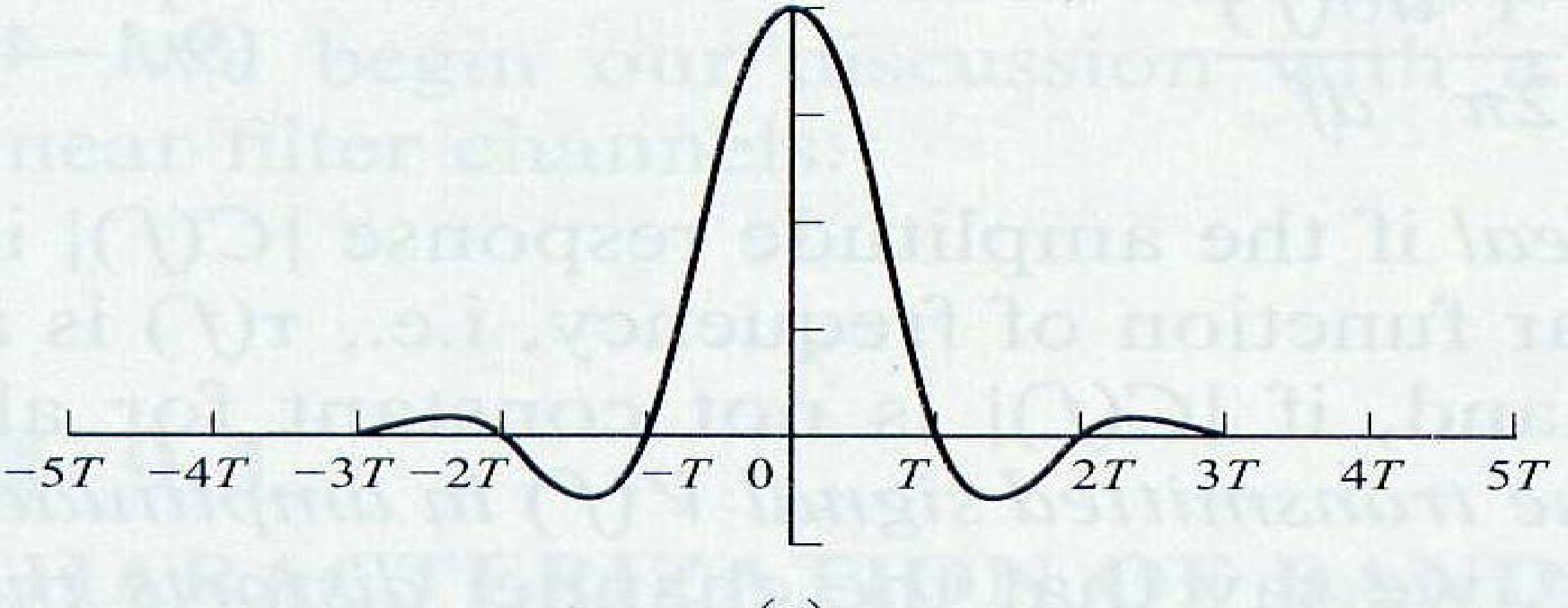
- However, transmission of the pulse through a channel modeled as having a linear envelope delay τ(*f*) [quadratic phase θ(*f*)] results in the received pulse shown in Fig. (b), where the zero-crossings that are no longer periodically spaced.



(b)

Characterization of Band-Limited Channels

- A sequence of successive pulses would no longer be distinguishable. Thus, the channel delay distortion results in ISI.
- It is possible to compensate for the nonideal frequency-response of the channel by use of a *filter* or *equalizer* at the demodulator.
- Fig. (c) illustrates the output of a linear equalizer that compensates for the linear distortion in the channel.



(c)

Characterization of Band-Limited Channels

- The equivalent low-pass transmitted signal for several different types of digital modulation techniques has the common form

$$v(t) = \sum_{n=0}^{\infty} I_n g(t - nT)$$

where $\{I_n\}$: discrete information-bearing sequence of symbols.

$g(t)$: a pulse with band-limited frequency-response $G(f)$, i.e., $G(f) = 0$ for $|f| > W$.

- This signal is transmitted over a channel having a frequency response $C(f)$, also limited to $|f| \leq W$.
- The received signal can be represented as

$$r_l(t) = \sum_{n=0}^{\infty} I_n h(t - nT) + z(t)$$

where $h(t) = \int_{-\infty}^{\infty} g(\tau) c(t - \tau) d\tau$ and $z(t)$ is the AWGN.

Signal Design for Band-Limited Channels

- Suppose that the received signal is passed first through a filter and then sampled at a rate 1/*T* samples/s, the optimum filter from the point of view of signal detection is one matched to the received pulse. That is, the frequency response of the receiving filter is $H^*(f)$.
- We denote the output of the receiving filter as

$$y(t) = \sum_{n=0}^{\infty} I_n x(t - nT) + v(t)$$

where

$x(t)$: the pulse representing the response of the receiving filter to the input pulse $h(t)$.

$v(t)$: response of the receiving filter to the noise $z(t)$.

Signal Design for Band-Limited Channels

- If $y(t)$ is sampled at times $t = kT + \tau_0$, $k = 0, 1, \cdots$, we have

$$y(kT+\tau_0) \equiv y_k = \sum_{n=0}^{\infty} I_n x(kT - nT + \tau_0) + v(kT+\tau_0)$$

or, equivalently,

$$y_k = \sum_{n=0}^{\infty} I_n x_{k-n} + v_k, \quad k = 0,1,\ldots$$

where τ_0: transmission delay through the channel.

- The sample values can be expressed as

$$y_k = x_0 \left(I_k + \frac{1}{x_0} \sum_{\substack{n=0 \\ n\neq k}}^{\infty} I_n x_{k-n} \right) + v_k, \quad k = 0,1,\ldots$$

Signal Design for Band-Limited Channels

- We regard x_0 as an arbitrary scale factor, which we arbitrarily set equal to unity for convenience, then

$$y_k = I_k + \sum_{\substack{n=0 \\ n\neq k}}^{\infty} I_n x_{k-n} + v_k$$

where

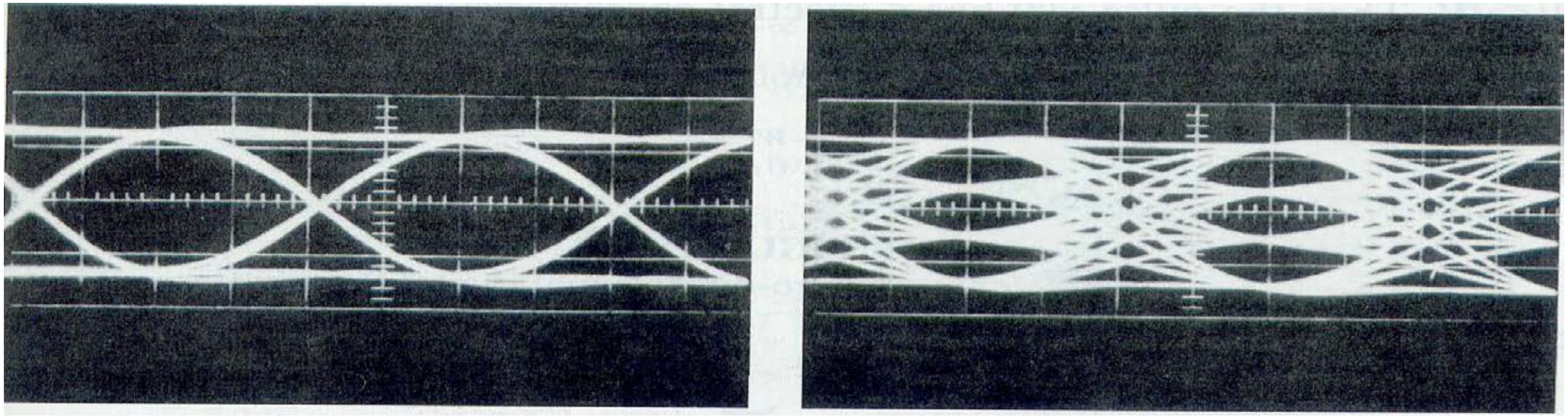
I_k: the desired information symbol at the k-th sampling instant.

$\sum_{\substack{n=0 \\ n\neq k}}^{\infty} I_n x_{k-n}$: ISI

v_k: additive Gaussian noise variable at the k-th sampling instant.

Signal Design for Band-Limited Channels

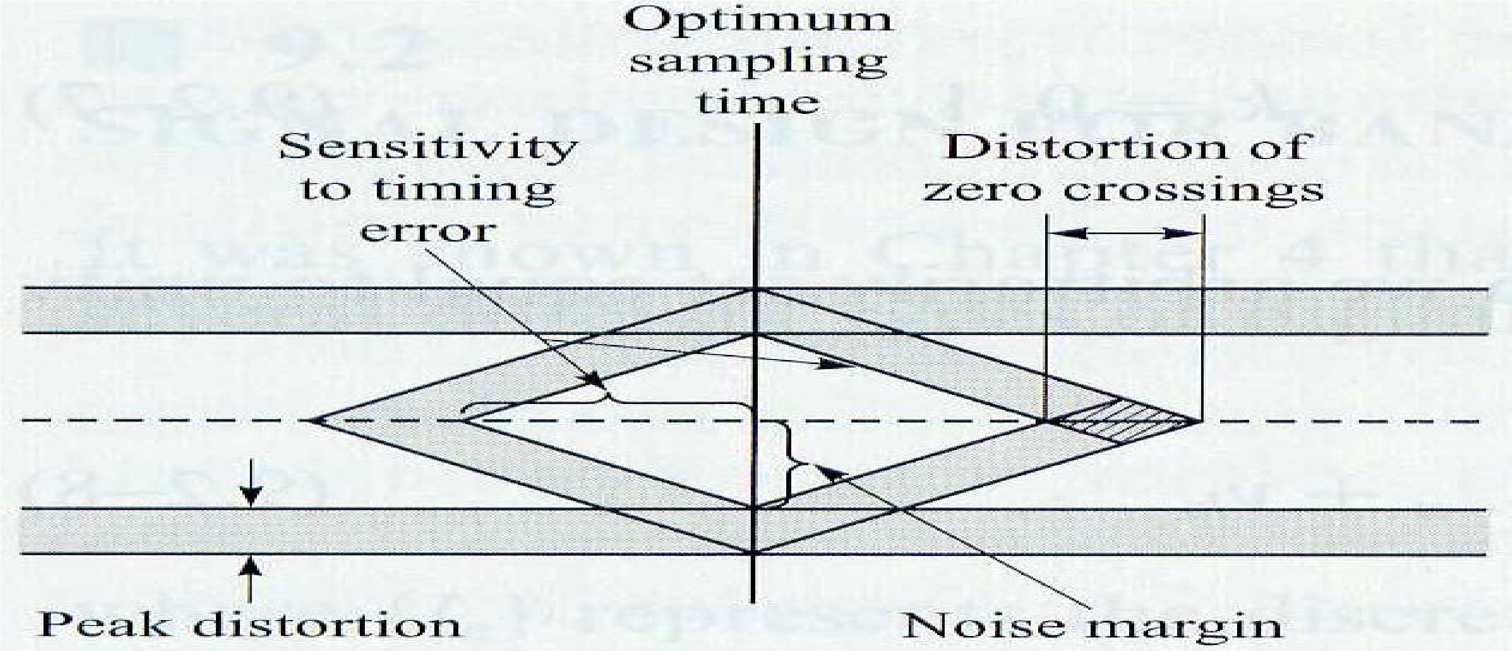
- The amount of ISI and noise in a digital communication system can be viewed on an oscilloscope.
- For PAM signals, we can display the received signal $y(t)$ on the vertical input with the horizontal sweep rate set at $1/T$.
- The resulting oscilloscope display is called an *eye pattern*.
- Eye patterns for binary and quaternary amplitude-shift keying:



  - The effect of ISI is to cause the eye to close.
  - Thereby, reducing the margin for additive noise to cause errors.

Signal Design for Band-Limited Channels

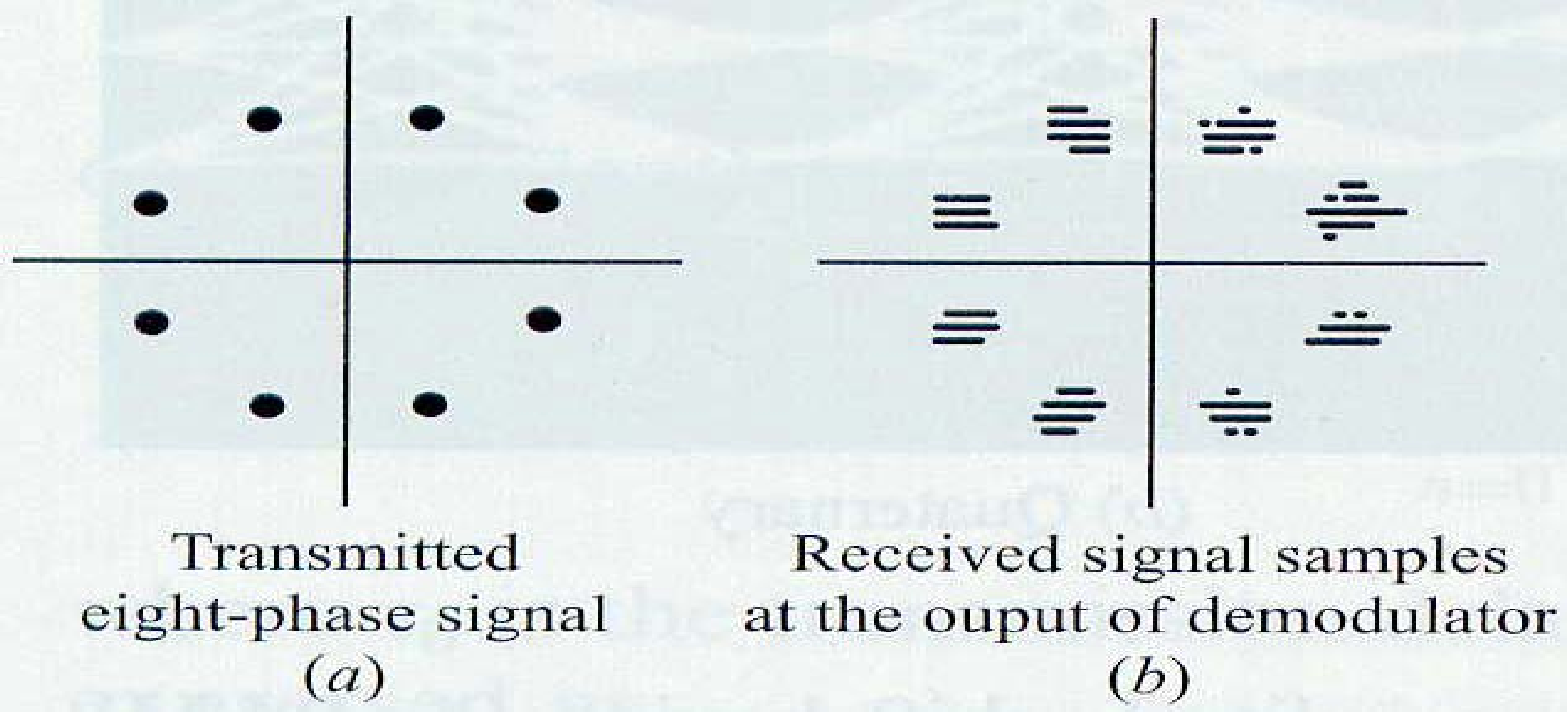
- Effect of ISI on eye opening:



  - ISI distorts the position of the zero-crossings and causes a reduction in the eye opening.
  - Thus, it causes the system to be more sensitive to a synchronization error.

Signal Design for Band-Limited Channels

- For PSK and QAM, it is customary to display the “eye pattern” as a two-dimensional scatter diagram illustrating the sampled values $\{y_k\}$ that represent the decision variables at the sampling instants.
- Two-dimensional digital “eye patterns.”



Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- Assuming that the band-limited channel has ideal frequency-response, i.e., $C(f) = 1$ for $|f| \leq W$, then the pulse $x(t)$ has a spectral characteristic $X(f) = |G(f)|^2$, where

$$x(t) = \int_{-W}^{W} X(f) e^{j2\pi ft} df$$

- We are interested in determining the spectral properties of the pulse $x(t)$, that results in no inter-symbol interference.
- Since

$$y_k = I_k + \sum_{\substack{n=0 \\ n\neq k}}^{\infty} I_n x_{k-n} + v_k$$

the condition for no ISI is

$$x(t = kT) \equiv x_k = \begin{cases} 1 & (k=0) \\ 0 & (k \neq 0) \end{cases} \qquad (*)$$

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- Nyquist pulse-shaping criterion (Nyquist condition for zero ISI)
  - The necessary and sufficient condition for $x(t)$ to satisfy

$$x(nT)=\begin{cases}1 & (n=0)\\ 0 & (n\neq 0)\end{cases}$$

is that its Fourier transform $X(f)$ satisfy

$$\sum_{m=-\infty}^{\infty} X(f+m/T)=T$$

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- Proof:
  - In general, $x(t)$ is the inverse Fourier transform of $X(f)$. Hence,

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

  - At the sampling instant $t = nT$,

$$x(nT) = \int_{-\infty}^{\infty} X(f) e^{j2\pi fnT} df$$

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- Breaking up the integral into integrals covering the finite range of 1/*T*, thus, we obtain

$$x(nT)=\sum_{m=-\infty}^{\infty}\int_{(2m-1)/2T}^{(2m+1)/2T}X(f)e^{j2\pi fnT}df$$

$$=\sum_{m=-\infty}^{\infty}\int_{-1/2T}^{1/2T}X(f'+m/T)e^{j2\pi f'nT}df'$$

$$=\int_{-1/2T}^{1/2T}\left[\sum_{m=-\infty}^{\infty}X(f+m/T)\right]e^{j2\pi fnT}df$$

$$=\int_{-1/2T}^{1/2T}B(f)e^{j2\pi fnT}df \qquad (1)$$

$$f=f'+\frac{m}{T}$$

$$f:\left[\frac{2m-1}{2T},\frac{2m+1}{2T}\right]$$

$$f':\left[\frac{-1}{2T},\frac{1}{2T}\right]$$

$$df=df'$$

where we define $B(f)$ as $B(f)=\sum_{m=-\infty}^{\infty}X(f+m/T)$

Periodic function.

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- Obviously $B(f)$ is a periodic function with period $1/T$, and, therefore, it can be expanded in terms of its Fourier series coefficients $\{b_n\}$ as

$$B\left(f\right)=\sum_{n=-\infty}^{\infty} b_n e^{j2\pi nfT} \qquad (2)$$

where

$$b_n = T\int_{-1/2T}^{1/2T} B\left(f\right)e^{-2\pi nfT}df \qquad (3)$$

$$x\left(nT\right)=\int_{-1/2T}^{1/2T} B\left(f\right)e^{j2\pi fnT}df \qquad (1)$$

- Comparing (1) and (3), we obtain

$$x\left(-nT\right)=\int_{-1/2T}^{1/2T} B\left(f\right)e^{-j2\pi fnT}df = \frac{1}{T}b_n$$

$$b_n = Tx\left(-nT\right) \qquad (4)$$

Recall that the conditions for no ISI are (from $*$):

$$x\left(t=kT\right)\equiv x_k = \begin{cases} 1 & \left(k=0\right) \\ 0 & \left(k\neq 0\right) \end{cases}$$

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- Therefore, the necessary and sufficient condition for

$$y_k = I_k + \sum_{\substack{n=0 \\ n\neq k}}^{\infty} I_n x_{k-n} + v_k$$

to be satisfied is that

$$b_n = \begin{cases} T & (n=0) \\ 0 & (n\neq 0) \end{cases}$$

which, when substituted into $B(f) = \sum_{n=-\infty}^{\infty} b_n e^{j2\pi nfT}$, yields

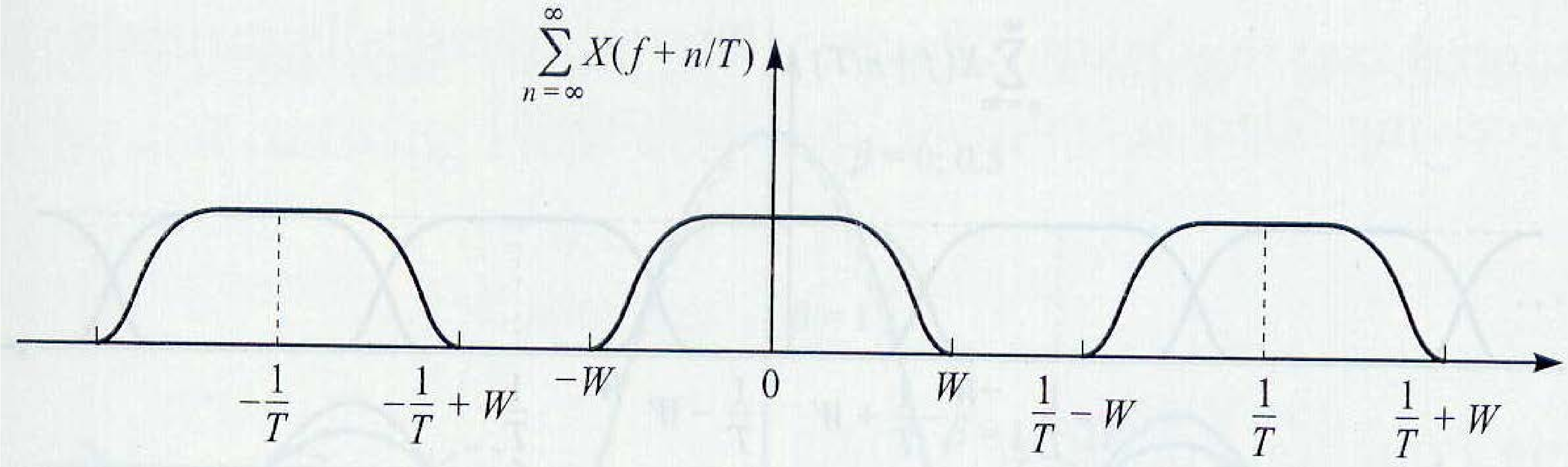
$$B(f) = T$$

or, equivalently $\sum_{m=-\infty}^{\infty} X(f + m/T) = T$

This concludes the proof of the theorem.

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

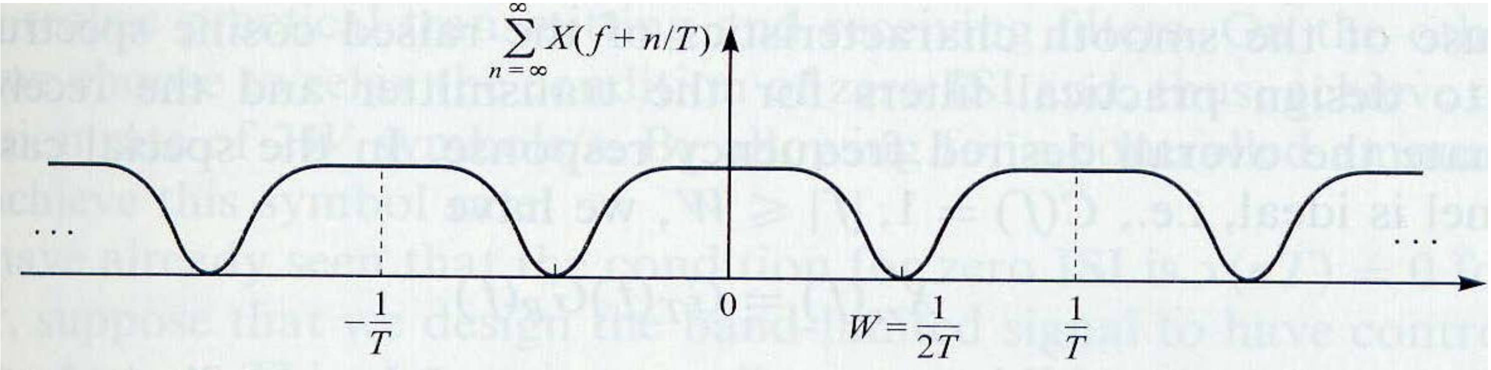
- Suppose that the channel has a bandwidth of W. Then $C(f) \equiv 0$ for $|f| > W$ and $X(f) = 0$ for $|f| > W$.
  - When $T < 1/2W$ (or $1/T > 2W$)



  - Since $B(f) = \sum_{-\infty}^{+\infty} X\left(f + n/T\right)$ consists of nonoverlapping replicas of $X(f)$, separated by $1/T$, there is no choice for $X(f)$ to ensure $B(f) \equiv T$ in this case and there is no way that we can design a system with no ISI.

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- When $T = 1/2W$, or $1/T = 2W$ (the Nyquist rate), the replications of $X(f)$, separated by $1/T$, are shown below:



- In this case, there exists only one $X(f)$ that results in $B(f) = T$, namely,

$$X(f)=\begin{cases} T & \left(|f|<W\right) \\ 0 & \left(otherwise\right) \end{cases}$$

which corresponds to the pulse

$$x(t)=\frac{\sin\left(\pi t/T\right)}{\pi t/T}\equiv \operatorname{sinc}\left(\frac{\pi t}{T}\right)$$

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

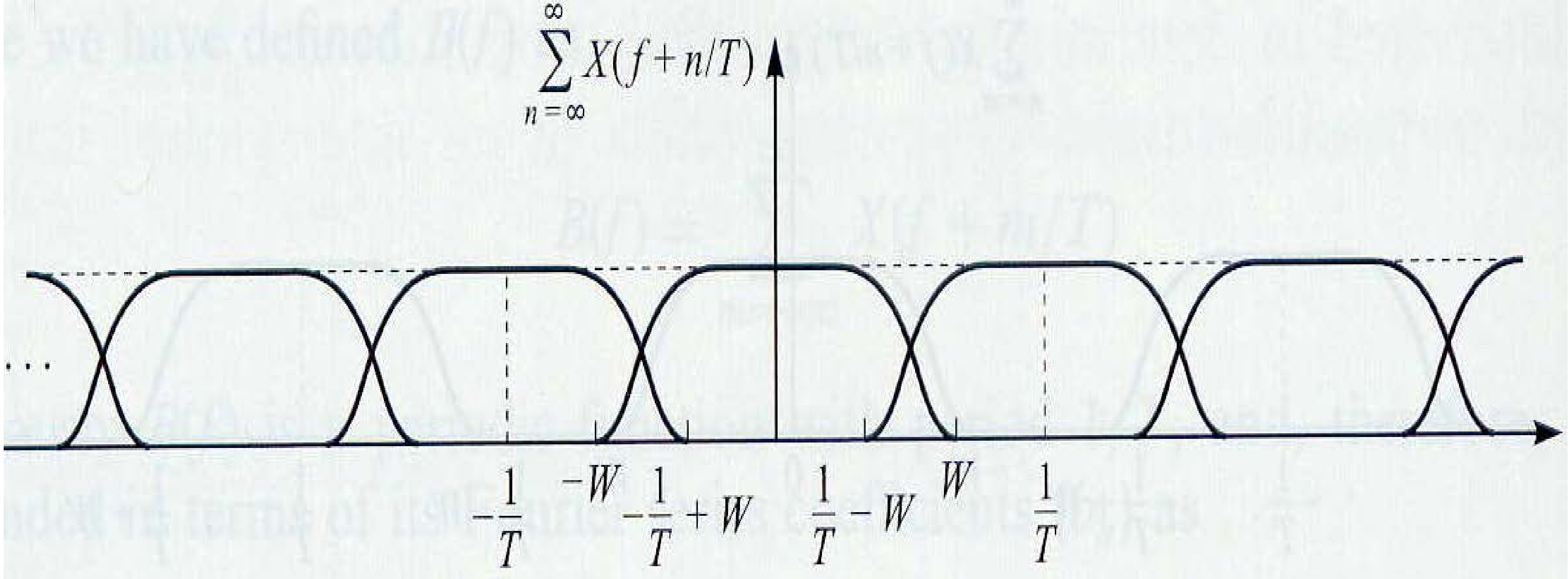
- The smallest value of T for which transmission with zero ISI is possible is $T = 1/2W$, and for this value, $x(t)$ has to be a sinc function.
- The difficulty with this choice of $x(t)$ is that it is noncausal and nonrealizable.
- A second difficulty with this pulse shape is that its rate of convergence to zero is slow.
- The tails of $x(t)$ decay as $1/t$; consequently, a small mistiming error in sampling the output of the matched filter at the demodulator results in an infinite series of ISI components.
- Such a series is not absolutely summable because of the $1/t$ rate of decay of the pulse, and, hence, the sum of the resulting ISI does not converge.

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=?$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=?$$

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- When $T > 1/2W$ (or $1/T<2W$), $B(f)$ consists of overlapping replications of $X(f)$ separated by $1/T$:



- In this case, there exist numerous choices for $X(f)$ such that $B(f) \equiv T$.

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- A particular pulse spectrum, for the $T > 1/2W$ case, that has desirable spectral properties and has been widely used in practice is the *raised cosine spectrum*.
- Raised cosine spectrum:

$$X_{rc}(f)=\begin{cases} T & \left(0\le|f|\le\frac{1-\beta}{2T}\right) \\ \frac{T}{2}\left\{1+\cos\left[\frac{\pi T}{\beta}\left(|f|-\frac{1-\beta}{2T}\right)\right]\right\} & \left(\frac{1\text{-}\beta}{2\mathrm{T}}\le|f|\le\frac{1+\beta}{2T}\right) \\ 0 & \left(|f|>\frac{1+\beta}{2T}\right) \end{cases}$$

  - β: roll-off factor. $(0 \le \beta \le 1)$

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- The bandwidth occupied by the signal beyond the Nyquist frequency $1/2T$ is called the *excess bandwidth* and is usually expressed as a percentage of the Nyquist frequency.
  - β= 1/2 => excess bandwidth = 50 %.
  - β= 1 => excess bandwidth = 100%.
- The pulse $x(t)$, having the raised cosine spectrum, is

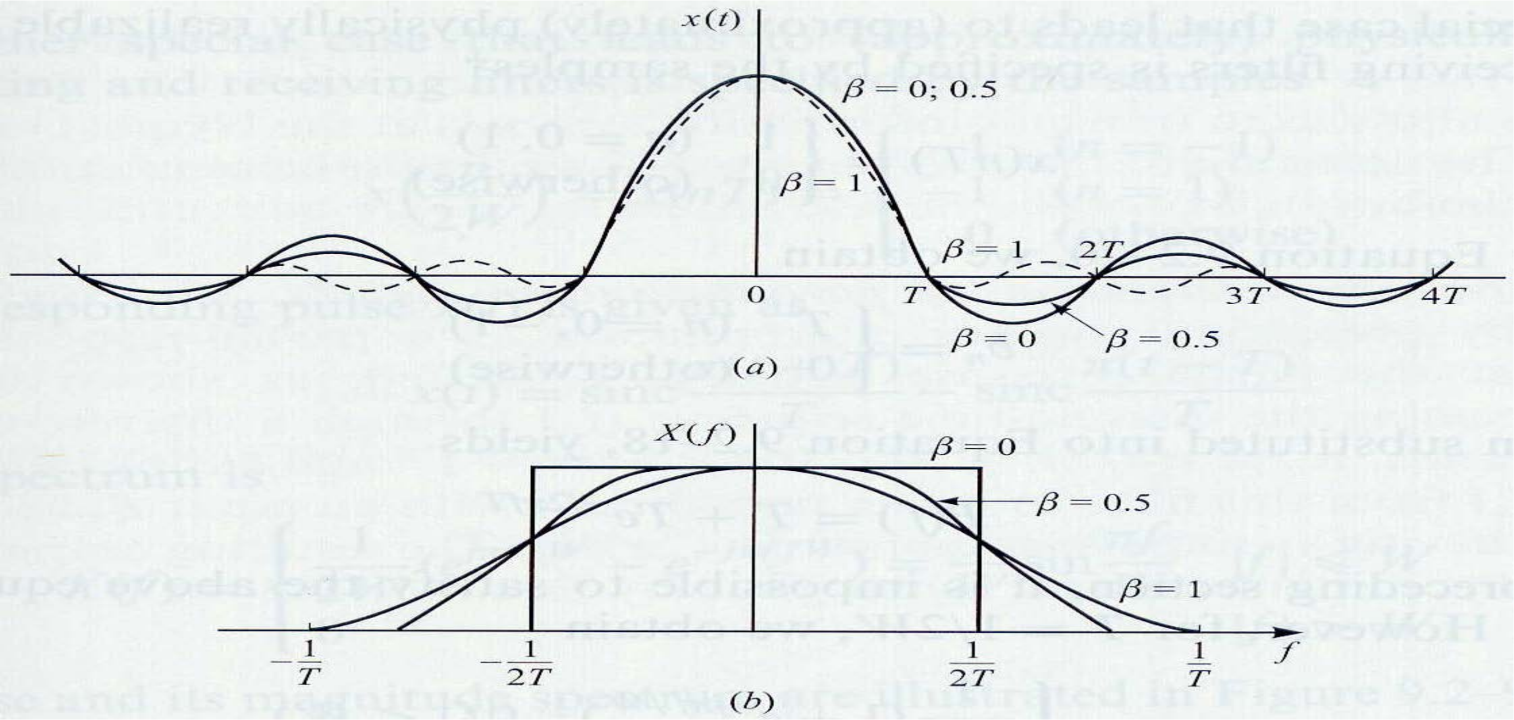
$$x(t) = \frac{\sin(\pi t/T)}{\pi t/T} \frac{\cos(\pi\beta t/T)}{1-4\beta^2 t^2/T^2}$$

$$= \sin c(\pi t/T) \frac{\cos(\pi\beta t/T)}{1-4\beta^2 t^2/T^2}$$

  - $x(t)$ is normalized so that $x(0) = 1$.

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

- Pulses having a raised cosine spectrum:



  - Forβ= 0, the pulse reduces to $x(t) = \mathrm{sinc}(\pi t/T)$, and the symbol rate $1/T = 2W$.
  - When β= 1, the symbol rate is $1/T = W$.

Design of Band-Limited Signals for No ISI
The Nyquist Criterion

  - In general, the tails of $x(t)$ decay as $1/t^3$ for $\beta > 0$.
  - Consequently, a mistiming error in sampling leads to a series of ISI components that converges to a finite value.
- Because of the smooth characteristics of the raised cosine spectrum, it is possible to design practical filters for the transmitter and the receiver that approximate the overall desired frequency response.
- In the special case where the channel is ideal, i.e., $C(f) = 1$, $|f| \leq W$, we have

$$X_{rc}(f) = G_T(f)G_R(f)$$

where $G_T(f)$ and $G_R(f)$ are the frequency responses of the two filters.

Design of Band-Limited Signals for No ISI The Nyquist Criterion

- If the receiver filter is matched to the transmitter filter, we have $X_{rc}(f) = G_T(f)\,G_R(f) = |G_T(f)|^2$. Ideally,

$$G_T(f) = \sqrt{\left|X_{rc}(f)\right|}\,e^{-j2\pi f t_0}$$

and $G_R(f) = G_T^*(f)$, where t_0 is some nominal delay that is required to ensure physical realizability of the filter.

- Thus, the overall raised cosine spectral characteristic is split evenly between the transmitting filter and the receiving filter.
- An additional delay is necessary to ensure the physical realizability of the receiving filter.

Square Root Raised Cosine Filter

- The cosine roll-off transfer function can be achieved by using identical square root raised cosine filter $\sqrt{X_{rc}(f)}$ at the transmitter and receiver.
- The pulse *SRRC*(*t*), having the square root raised cosine spectrum, is

$$SRRC(t)=\frac{\sin\left(\pi\frac{t}{T_C}(1-\beta)\right)+4\beta\frac{t}{T_C}\cos\left(\pi\frac{t}{T_C}(1+\beta)\right)}{\pi\frac{t}{T_C}\left(1-\left(4\beta\frac{t}{T_C}\right)^2\right)}$$

where T_C is the inverse of chip rate ($\approx 0.2604167\mu$s)

and β = 0.22 for WCDMA.

Design of Band-limited Signals with Controlled ISI -- Partial-Response Signals

- It is necessary to reduce the symbol rate *1/T* below the Nyquist rate of 2*W* symbols/s to realize practical transmitting and receiving filters.
- Suppose we choose to relax the condition of zero ISI and, thus, achieve a symbol transmission rate of 2*W* symbols/s.
- By allowing for a controlled amount of ISI, we can achieve this symbol rate.
- The condition for zero ISI is $x(nT)=0$ for $n\neq 0$.
- Suppose that we design the band-limited signal to have controlled ISI at one time instant. This means that we allow one additional nonzero value in the samples $\{x(nT)\}$.

Design of Band-limited Signals with Controlled ISI -- Partial-Response Signals

- One special case that leads to (approximately) physically realizable transmitting and receiving filters is the *duobinary signal pulse*:

$$x(nT)=\begin{cases}1 & (n=0,1)\\ 0 & (\text{otherwise})\end{cases}$$

  - Using Equation (4) in Page 17

$$b_n = Tx(-nT)$$

$$b_n = \begin{cases}T & (n=0,\text{-}1)\\ 0 & (\text{otherwise})\end{cases}$$

$$\boxed{B(f)=\sum_{n=-\infty}^{\infty} b_n e^{j2\pi nfT}}$$

  - When substituted into Equation (2) in Page 17, we obtain:

$$B(f)=T+Te^{-j2\pi fT}$$

Design of Band-limited Signals with Controlled ISI -- Partial-Response Signals

- It is impossible to satisfy the above equation for $1/T>2W$.
- For $T=1/2W$, we obtain

$$X(f)=\begin{cases}\frac{1}{2W}\left(1+e^{-j\pi f/W}\right) & \left(|f|<W\right)\\ 0 & (\text{otherwise})\end{cases}$$

$$=\begin{cases}\frac{1}{W}e^{-j\pi f/2W}\cos\frac{\pi f}{2W} & \left(|f|<W\right)\\ 0 & (\text{otherwise})\end{cases}$$

- Therefore, $x(t)$ is given by:

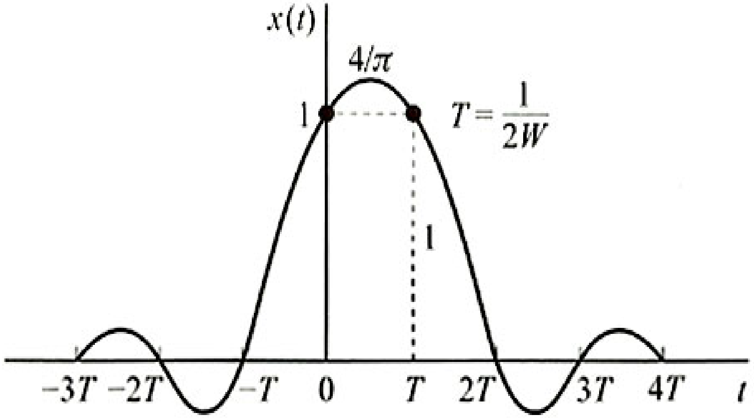
$$x(t)=\operatorname{sinc}(2\pi Wt)+\operatorname{sinc}\left[2\pi\left(Wt-\frac{1}{2}\right)\right]$$

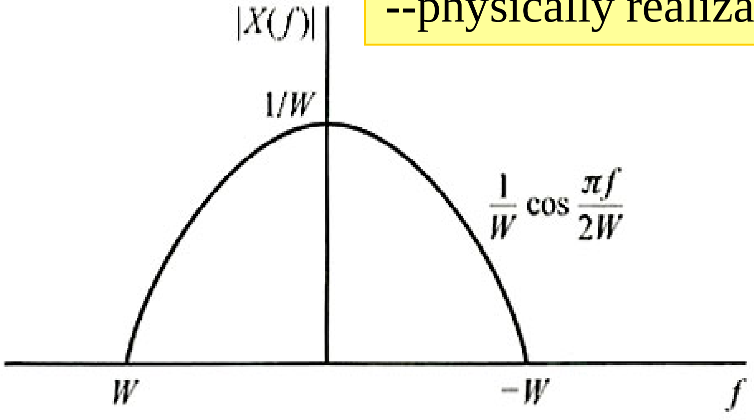
- This pulse is called a *duobinary signal pulse*.

Design of Band-limited Signals with Controlled ISI -- Partial-Response Signals

- Time-domain and frequency-domain characteristics of a duobinary signal.

The spectrum decays to zero smoothly.
--physically realizable





- *Modified duobinary signal pulse*:

$$x\left(\frac{n}{2W}\right) = x(nT) = \begin{cases} 1 & (n = \text{-}1) \\ -1 & (n = 1) \\ 0 & (\text{otherwise}) \end{cases}$$

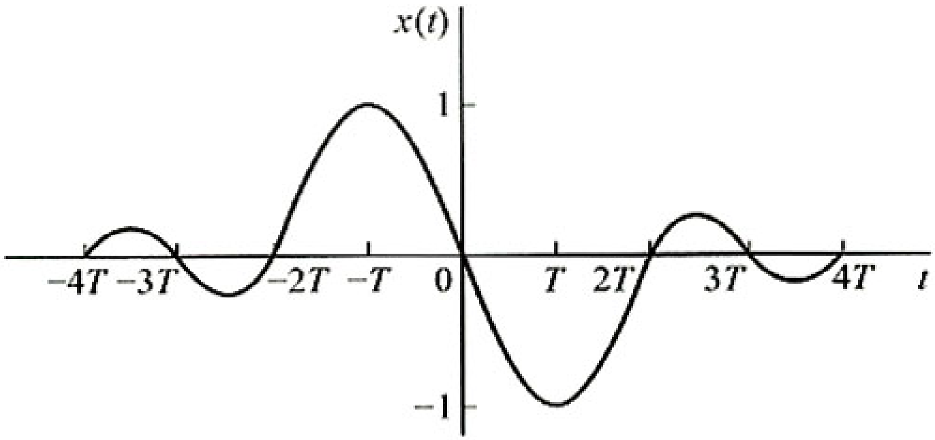
Design of Band-limited Signals with Controlled ISI -- Partial-Response Signals

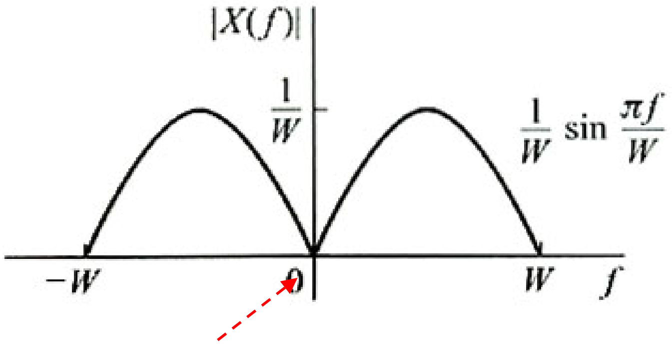
- The corresponding pulse $x(t)$ is given as

$$x(t) = \operatorname{sinc}\frac{\pi(t+T)}{T} - \operatorname{sinc}\frac{\pi(t-T)}{T}$$

- The spectrum is given by

$$X(f) = \begin{cases} \dfrac{1}{2W}\left(e^{j\pi f/W} - e^{-j\pi f/W}\right) = \dfrac{j}{W}\sin\dfrac{\pi f}{W} & |f| \le W \\ 0 & |f| > W \end{cases}$$





Zero at f=0.

Design of Band-limited Signals with Controlled ISI -- Partial-Response Signals

- Other physically realizable filter characteristics are obtained by selecting different values for the samples $\{x(n/2W)\}$ and more than two nonzero samples.
- As we select more nonzero samples, the problem of unraveling the controlled ISI becomes more cumbersome and impractical.
- When controlled ISI is purposely introduced by selecting two or more nonzero samples form the set $\{x(n/2W)\}$, the class of band-limited signal pulses are called *partial-response signals*:

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \operatorname{sinc}\left[2\pi W\left(t - \frac{n}{2W}\right)\right]$$

$$X(f) = \begin{cases} \frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-jn\pi f/W} & \left(|f| \le W\right) \\ 0 & \left(|f| > W\right) \end{cases}$$

Data Detection for Controlled ISI

- Two methods for detecting the information symbols at the receiver when the received signal contains controlled ISI:
  - Symbol-by-symbol detection method.
    - Relatively easy to implement.
  - Maximum-likelihood criterion for detecting a sequence of symbols.
    - Minimizes the probability of error but is a little more complex to implement.
- The following treatment is based on PAM signals, but it is easily generalized to QAM and PSK.
- We assume that the desired spectral characteristic $X(f)$ for the partial-response signal is split evenly between the transmitting and receiving filters, i.e., $|G_T(f)|=|G_R(f)|=|X(f)|^{1/2}$.

Data Detection for Controlled ISI

- Symbol-by-symbol suboptimum detection
  - For duobinary signal pulse, $x(nT)=1$, for $n=0,1$, and is zero otherwise.
  - The samples at the output of the receiving filter (demodulator) have the form

$$y_m = B_m + v_m = I_m + I_{m-1} + v_m$$

    where $\{I_m\}$ is the transmitted sequence of amplitudes and $\{v_m\}$ is a sequence of additive Gaussian noise samples.
  - Consider the binary case where $I_m=\pm1$, B_m takes on one of three possible values, namely, B_m =-2,0,2 with corresponding probabilities ¼, ½, ¼.
    - If I_{m-1} is the detected symbol from the (m-1)th signaling

Data Detection for Controlled ISI

- Symbol-by-symbol suboptimum detection

  interval, its effect on B_m, the received signal in the mth signaling interval, can be eliminated by subtraction, thus allowing I_m to be detected.

  - Major problem with this procedure is *error propagation*: if $I_{m\text{-}1}$ is in error, its effect on B_m is not eliminated but, in fact, is reinforced by the incorrect subtraction.

- Error propagation can be avoided by *precoding* the data.
  - The precoding is performed on the binary data sequence prior to modulation.
  - From the data sequence $\{D_n\}$, the precoded sequence $\{P_n\}$ is given by:

Modulo-2 subtraction

$$P_m = D_m \ominus P_{m-1}, \quad m = 1, 2, \ldots$$

Data Detection for Controlled ISI

- Symbol-by-symbol suboptimum detection
  - Set I_m=-1 if P_m=0 and I_m=1 if P_m=1, i.e., I_m=2P_m-1.
  - The noise-free samples at the output of the receiving filter are given by

$$B_m = I_m + I_{m-1} = \left(2P_m - 1\right) + \left(2P_{m-1} - 1\right) = 2\left(P_m + P_{m-1} - 1\right)$$

$$P_m + P_{m-1} = \frac{1}{2} B_m + 1$$

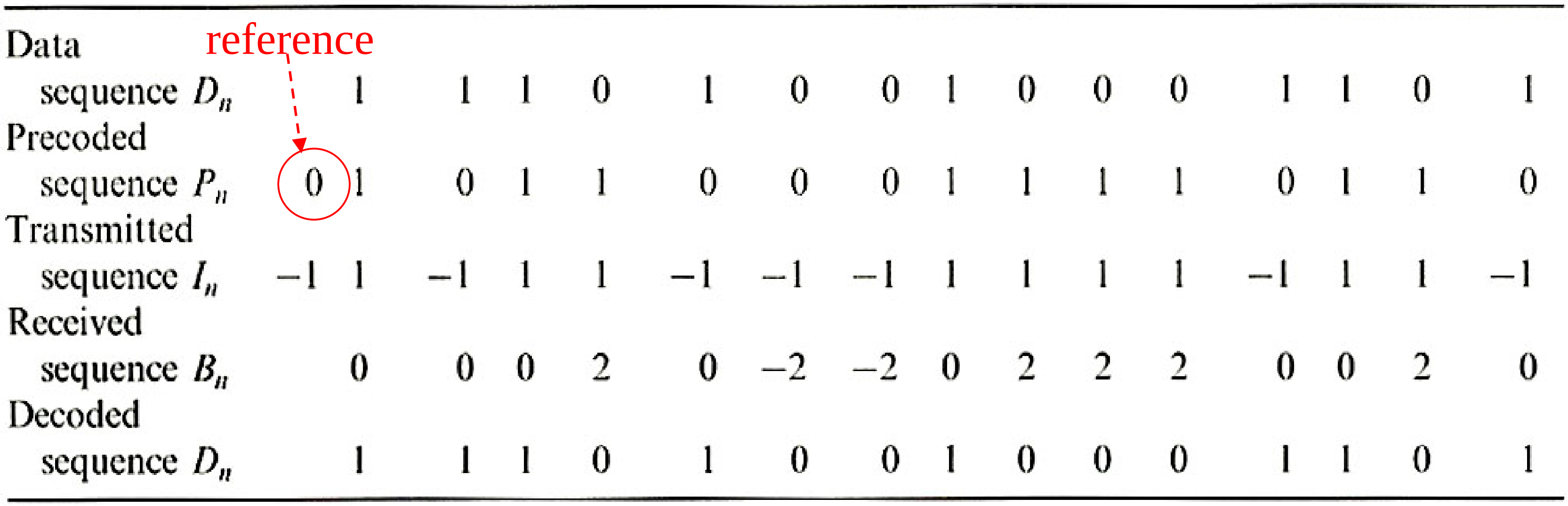
  - Since $D_m = P_m \oplus P_{m-1}$, it follows that the data sequence D_m is obtained from B_m using the relation:

$$D_m = \frac{1}{2} B_m + 1 \quad \left(\mathrm{mod}\ 2\right)$$

  - Consequently, if B_m=±2, then D_m=0, and if B_m=0, D_m=1.

Data Detection for Controlled ISI

- Symbol-by-symbol suboptimum detection
  - Binary signaling with duobinary pulses



reference

Data sequence D_n		1	1	1	0	1	0	0	1	0	0	0	1	1	0	1
Precoded sequence P_n	0	1	0	1	1	0	0	0	1	1	1	1	0	1	1	0
Transmitted sequence I_n	−1	1	−1	1	1	−1	−1	−1	1	1	1	1	−1	1	1	−1
Received sequence B_n		0	0	0	2	0	−2	−2	0	2	2	2	0	0	2	0
Decoded sequence D_n		1	1	1	0	1	0	0	1	0	0	0	1	1	0	1

- The extension from binary PAM to multilevel PAM signaling
  - The M-level amplitude sequence $\{I_m\}$ results in a noise-free sequence

$$B_m = I_m + I_{m-1}, \quad m = 1, 2, \ldots$$

Data Detection for Controlled ISI

- Symbol-by-symbol suboptimum detection

  which has 2M-1 possible equally spaced levels.

  - The amplitude levels are determined from the relation:

  $$I_m = 2P_m - (M-1)$$

  where $\{P_m\}$ is the precoded sequence that is obtained from an M-level data sequence $\{D_m\}$ according to the relation

  $$P_m = D_m \ominus P_{m-1} \quad (\text{mod } M)$$

  where the possible values of the sequence $\{D_m\}$ are 0, 1, 2, ⋯, M-1.

  - In the absence of noise, the samples at the output is given by:

  $$B_m = I_m + I_{m-1} = 2[P_m + P_{m-1} - (M-1)]$$

Data Detection for Controlled ISI

- Symbol-by-symbol suboptimum detection
  - Hence $P_m + P_{m-1} = \frac{1}{2} B_m + (M-1)$
  - Since $D_m = P_m + P_{m-1}$ (mod M), it follows that

$$D_m = \frac{1}{2} B_m + (M-1) \quad (\text{mod } M)$$

  - Four-level signal transmission with duobinary pulses

(M=4)

Data sequence D_m		0	0	1	3	1	2	0	3	3	2	0	1	0
Precoded sequence P_m	0	0	0	1	2	3	3	1	2	1	1	3	2	2
Transmitted sequence I_m	−3	−3	−3	−1	1	3	3	−1	1	−1	−1	3	1	1
Received sequence B_m		−6	−6	−4	0	4	6	2	0	0	−2	2	4	2
Decoded sequence D_m		0	0	1	3	1	2	0	3	3	2	0	1	0

Data Detection for Controlled ISI

- Symbol-by-symbol suboptimum detection
  - In the case of the modified duobinary pulse, the controlled ISI is specified by the values $x(n/2W)$=-1, for n=1, $x(n/2W)$=1, for n=-1, and zero otherwise.
    - The noise-free sampled output from the receiving filter is given as:

$$B_m = I_m - I_{m-2}$$

    - Where the M-level sequence $\{I_m\}$ is obtained by mapping a precoded sequence according to

$$I_m = 2P_m - (M-1)$$

and

$$P_m = D_m \oplus P_{m-2} \quad (\mathrm{mod}\ \ M)$$

Data Detection for Controlled ISI

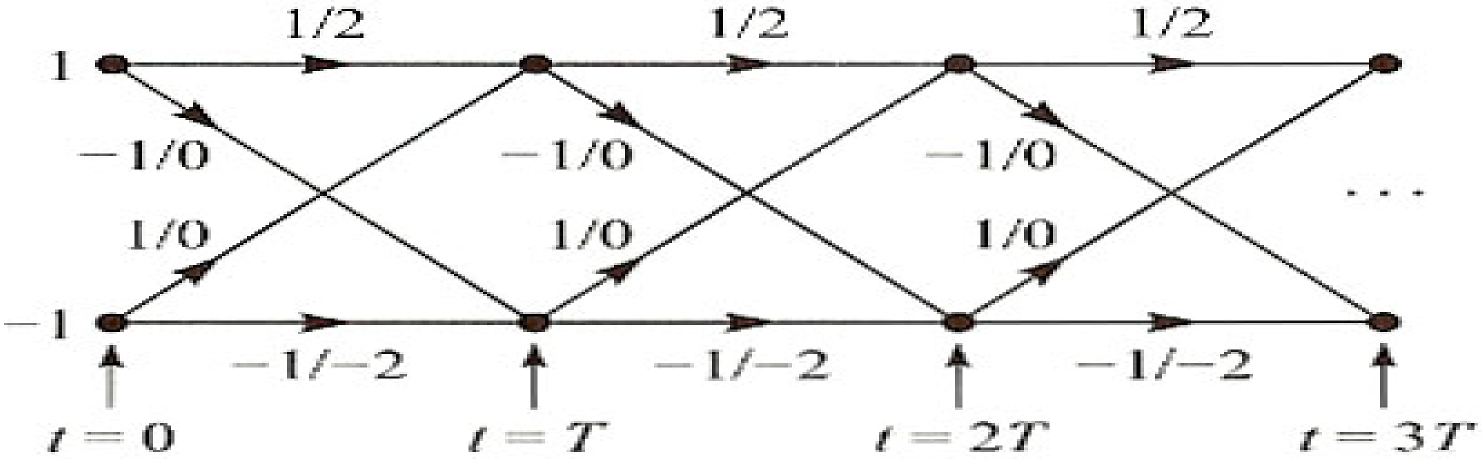
- Symbol-by-symbol suboptimum detection
  - From these relations, it is easy to show that the detection rule for recovering the data sequence $\{D_m\}$ from $\{B_m\}$ in the absence of noise is

$$D_m = \frac{1}{2} B_m \quad \left(\bmod\ M\right)$$

  - The precoding of the data at the transmitter makes it possible to detect the received data on a symbol-by-symbol basis without having to look back at previously detected symbols. Thus, error propagation is avoided.
  - The symbol-by-symbol detection rule is not the optimum detection scheme for partial-response signals. Nevertheless, it is relatively simple to implement.

Data Detection for Controlled ISI

- Maximum-likelihood Sequence Detection
  - Partial-response waveforms are signal waveforms with memory. This memory is conveniently represented by a trellis.
  - The trellis for the duobinary partial-response signal for binary data transmission is illustrated in the following figure.



    - The first number on the left is the new data bit and the number on the right is the received signal level.

Data Detection for Controlled ISI

- Maximum-likelihood Sequence Detection
  - The duobinary signal has a memory of length L=1. In general, for M-ary modulation, the number of trellis states is M^L.
  - The optimum maximum-likelihood sequence detector selects the most probable path through the trellis upon observing the received data sequence $\{y_m\}$ at the sampling instants $t=mT$, $m=1,2,\cdots$.
  - The trellis search is performed by the Viterbi algorithm.
  - For the class of partial-response signals, the received sequence $\{y_m, 1\le m\le N\}$ is generally described statistically by the joint PDF $p(\mathbf{y}_N|\mathbf{I}_N)$, where $\mathbf{y}_N = [y_1\ y_2\ \cdots\ y_N]'$ and $\mathbf{I}_N = [I_1\ I_2\ \cdots\ I_N]'$ and $N>L$.

Data Detection for Controlled ISI

- Maximum-likelihood Sequence Detection
  - When the additive noise is zero-mean Gaussian, $p(\mathbf{y}_N|\mathbf{I}_N)$ is a multivariate Gaussian PDF, i.e.,

$$p\left(\mathbf{y}_N \mid \mathbf{I}_N\right) = \frac{1}{\left(2\pi \det \mathbf{C}\right)^{N/2}} \exp\left[-\frac{1}{2}\left(\mathbf{y}_N - \mathbf{B}_N\right)' \mathbf{C}^{-1}\left(\mathbf{y}_N - \mathbf{B}_N\right)\right]$$

    where $\mathbf{B}_N = [B_1\ B_2\ \cdots\ B_N]'$ is the mean of the vector $\mathbf{y}_N$ and $\mathbf{C}$ is the N X N covariance matrix of $\mathbf{y}_N$.

  - The ML sequence detector selects the sequence through the trellis that maximizes the PDF $p(\mathbf{y}_N|\mathbf{I}_N)$.
  - Taking the natural logarithms of $p(\mathbf{y}_N|\mathbf{I}_N)$:

$$\ln p\left(\mathbf{y}_N \mid \mathbf{I}_N\right) = -\frac{1}{2} N \ln\left(2\pi \det \mathbf{C}\right) - \frac{1}{2}\left(\mathbf{y}_N - \mathbf{B}_N\right)' \mathbf{C}^{-1}\left(\mathbf{y}_N - \mathbf{B}_N\right)$$

Data Detection for Controlled ISI

- Maximum-likelihood Sequence Detection
  - Given the received sequence $\{y_m\}$, the data sequence $\{I_m\}$ that maximizes ln $p(\mathbf{y}_N|\mathbf{I}_N)$ is identical to the sequence $\{I_N\}$ that minimizes $(\mathbf{y}_N-\mathbf{B}_N)' \mathbf{C}^{-1}(\mathbf{y}_N-\mathbf{B}_N)$, i.e.,

$$\hat{\mathbf{I}}_N = \arg\min_{\mathbf{I}_N} \left[\left(\mathbf{y}_N - \mathbf{B}_N\right)' \mathbf{C}^{-1} \left(\mathbf{y}_N - \mathbf{B}_N\right) \right]$$

  - The metric computations in the trellis search are complicated by the correlation of the noise samples at the output of the matched filter for the partial-response signal.
  - In the case of the duobinary signal waveform, the correlation of the noise sequence $\{v_m\}$ is over two successive signal samples.

Data Detection for Controlled ISI

- Maximum-likelihood Sequence Detection
  - Hence, v_m and v_{m+k} are correlated for k=1 and uncorrelated for k>1.
  - If we isgnore the noise correlation by assuming that $E(v_m v_{m+k})=0$ for k>0, the computation can be simplified to

$$\hat{\mathbf{I}}_N = \arg\min_{\mathbf{I}_N}\left[\left(\mathbf{y}_N - \mathbf{B}_N\right)'\left(\mathbf{y}_N - \mathbf{B}_N\right)\right] = \arg\min_{\mathbf{I}_N}\left[\sum_{m=1}^{N}\left(y_m - \sum_{k=0}^{L} x_k I_{m-k}\right)^2\right]$$

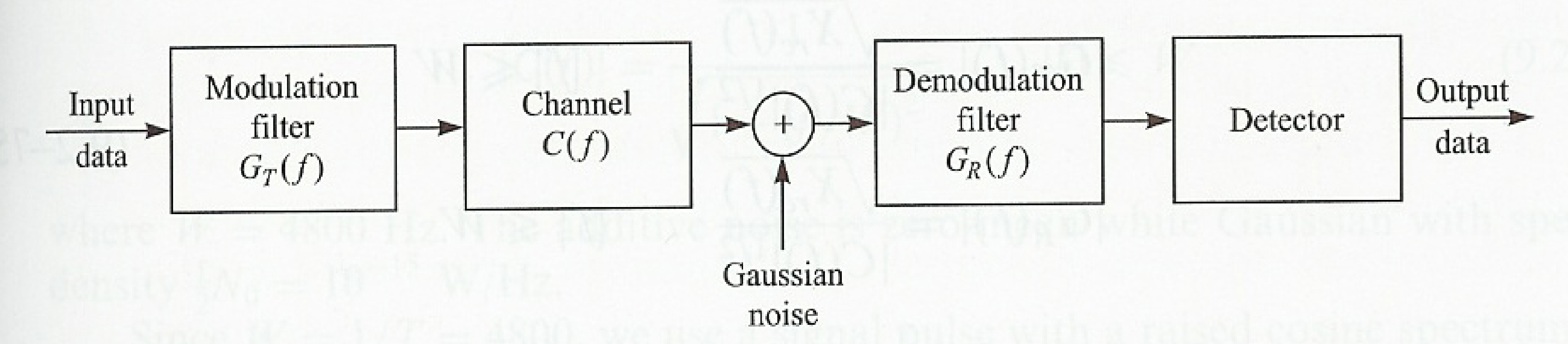
where

$$B_m = \sum_{k=0}^{L} x_k I_{m-k}$$

and $x_k = x(kT)$ are the sampled values of the partial-response signal waveform.

Signal Design for Channels with Distortion

- In this section, we perform the signal design under the condition that the channel distorts the transmitted signal.
- We assume that the channel frequency-response $C(f)$ is know for $|f| \leq W$ and that $C(f)=0$ for $|f|>W$.
- The filter responses $G_T(f)$ and $G_R(f)$ may be selected to minimize the error probability at the detector.
- The additive channel noise is assumed to be Gaussian with power spectral density $\Phi_{nn}(f)$.



Signal Design for Channels with Distortion

- For the signal component at the output of the demodulator, we must satisfy the condition

$$G_T(f)C(f)G_R(f) = X_d(f)e^{-j2\pi f t_0}, \quad |f| \le W$$

where $X_d(f)$ is the desired frequency response of the casecade of the modulator, channel, and demodulator, and t_0 is a time delay that is necessary to ensure the physical realizability of the modulation and demodulation filter.

- The desired frequency response $X_d(f)$ may be selected to yield either zero ISI or controlled ISI at the sampling instants.
- We shall consider the case of zero ISI by selecting $X_d(f) = X_{rc}(f)$, where $X_{rc}(f)$ is the raised cosine spectrum with an arbitrary roll-off factor.

Signal Design for Channels with Distortion

- The noise at the output of the demodulation filter may be expressed as

$$v(t)=\int_{-\infty}^{\infty} n(t-\tau)\,g_R(\tau)\,d\tau$$

where $n(t)$ is the input to the filter.

- Since $n(t)$ is zero-mean Gaussian, $v(t)$ is zero-mean Gaussian, with a power spectral density

$$\Phi_{vv}(f)=\Phi_{nn}(f)\left|G_R(f)\right|^2$$

- For simplicity, we consider binary PAM transmission. Then, the sampled output of the matched filter is

$$y_m = x_0 I_m + v_m = I_m + v_m$$

where x_0 is normalized to unity, $I_m=\pm d$, and v_m represents the noise term.

Signal Design for Channels with Distortion

- v_m is zero-mean Gaussian with variance

$$\sigma_v^2 = \int_{-\infty}^{\infty} \Phi_{nn}(f) \left| G_R(f) \right|^2 df$$

- The error probability is given by

SNR

$$P_2 = \frac{1}{\sqrt{2\pi}} \int_{d/\sigma_v}^{\infty} e^{-y^2/2} dy = Q\left(\sqrt{\frac{d^2}{\sigma_v^2}} \right)$$

- The probability of error is minimized by maximizing the ratio d^2/σ_v^2 .
- There are two possible solutions for the case in which the additive Gaussian noise is white so that $\Phi_{nn}(f) = N_0/2$.
- 1st solution: pre-compensate for the total channel distortion at the transmitter, so that the filter at the receiver is matched to the received signal.

Signal Design for Channels with Distortion

- The transmitter and receiver filters have the magnitude characteristics

$$\left|G_T(f)\right| = \frac{\sqrt{X_{rc}(f)}}{\left|C(f)\right|}, \quad |f| \leq W$$

$$\left|G_R(f)\right| = \sqrt{X_{rc}(f)}, \quad |f| \leq W \qquad (5)$$

- The phase characteristic of the channel frequency response $C(f)$ may also be compensated at the transmitter filter.
- For these filter characteristics, the average transmitted power is

$$P_{av} = \frac{E\left(I_m^2\right)}{T}\int_{-W}^{W} g_T^2(t)\,dt = \frac{d^2}{T}\int_{-W}^{W}\left|G_T(f)\right|^2 df = \frac{d^2}{T}\int_{-W}^{W}\frac{X_{rc}(f)}{\left|C(f)\right|^2}df$$

Signal Design for Channels with Distortion

- Hence,

$$d^2 = P_{av}T\left[\int_{-W}^{W}\frac{X_{rc}(f)}{|C(f)|^2}df\right]^{-1} \qquad (6)$$

- The noise at the output of the receiver filter is $\sigma^2_v = N_0/2$ and, hence, the SNR at the detector is

$$\frac{d^2}{\sigma_v^2} = \frac{2P_{av}T}{N_0}\left[\int_{-W}^{W}\frac{X_{rc}(f)}{|C(f)|^2}df\right]^{-1}$$

Signal Design for Channels with Distortion

- 2nd solution: As an alternative, suppose we split the channel compensation equally between the transmitter and receiver filters, i.e.,

$$|G_T(f)| = \frac{\sqrt{X_{rc}(f)}}{|C(f)|^{1/2}}, \quad |f| \le W$$

$$|G_R(f)| = \frac{\sqrt{X_{rc}(f)}}{|C(f)|^{1/2}}, \quad |f| \le W \qquad (7)$$

  - The phase characteristic of $C(f)$ may also be split equally between the transmitter and receiver filter.
  - The average transmitter power is

$$P_{av} = \frac{d^2}{T} \int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|} df$$

Signal Design for Channels with Distortion

- The noise variance at the output of the receiver filter is

$$\sigma_v^2 = \frac{N_0}{2} \int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|} df$$

- The SNR at the detector is

$$\frac{d^2}{\sigma_v^2} = \frac{2P_{av}T}{N_0} \left[\int_{-W}^{W} \frac{X_{rc}(f)}{|C(f)|} df \right]^{-2} \qquad (8)$$

- From Equations (6) (P.54) and Equation (8), we observe that when we express the SNR d^2/σ^2_v in terms of the average transmitter power P_{av}, there is a loss incurred due to channel distortion.

Signal Design for Channels with Distortion

- In the case of the filters given by Equation (5) (P.53), the loss is

$$10\log\int_{-W}^{W}\frac{X_{rc}(f)}{|C(f)|^2}df$$

- In the case of the filters given by Equation (7) (P.55) , the loss is

$$10\log\left[\int_{-W}^{W}\frac{X_{rc}(f)}{|C(f)|^2}df\right]^2$$

- When $C(f)=1$ for $|f|\leq W$, the channel is ideal and $\int_{-W}^{W}X_{rc}(f)df=1$ so that no loss is incurred.
- When there is amplitude distortion, $|C(f)|<1$ for some range of frequencies in the band $|f|\leq$W and there is a loss in SNR.
- It can be shown that the filters given by Equation (7) (P.55) result in the smaller SNR loss.