

Optimum Receivers for the Additive White Gaussian Noise Channel



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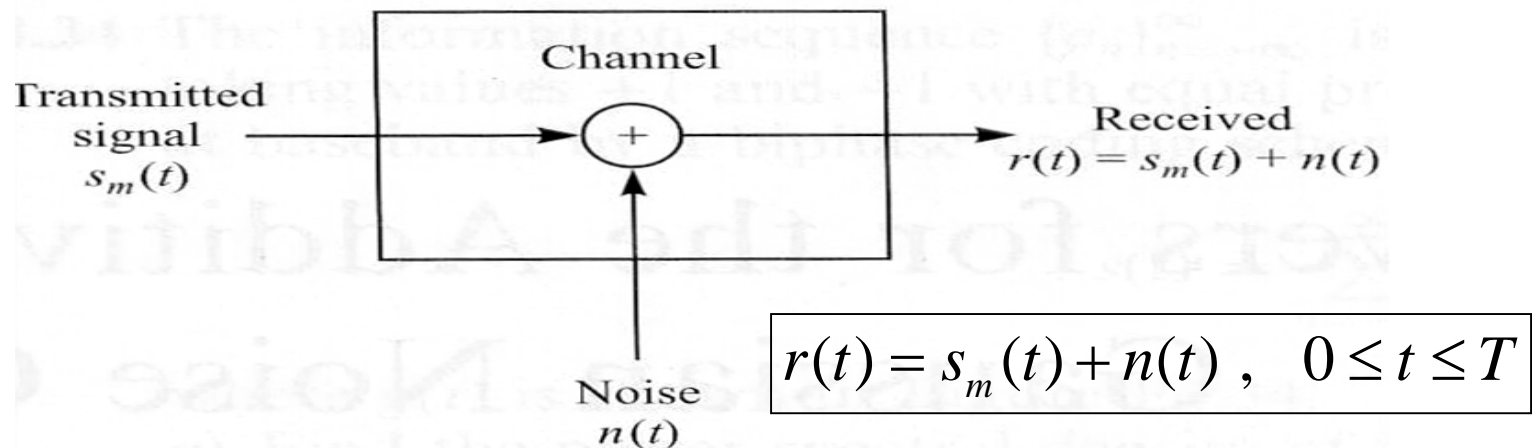


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Optimum Receiver for Signals Corrupted by Additive White Gaussian Noise



- ◇ We assume that the transmitter sends digital information by use of M signals waveforms $\{s_m(t)=1,2,\dots,M\}$. Each waveform is transmitted within the symbol interval of duration T , i.e. $0 \leq t \leq T$.
- ◇ The channel is assumed to corrupt the signal by the addition of white Gaussian noise, as shown in the following figure:

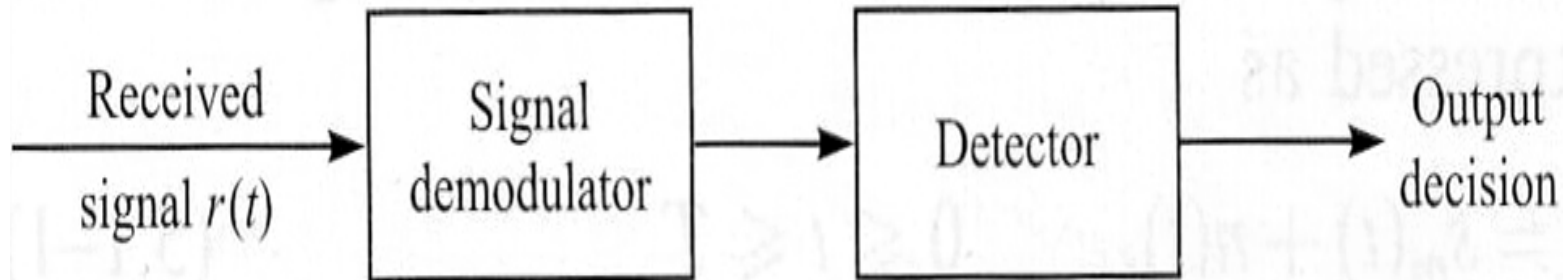


where $n(t)$ denotes a sample function of AWGN process with power spectral density $\Phi_{nn}(f) = \frac{1}{2}N_0$ W/Hz.

Optimum Receiver for Signals Corrupted by Additive White Gaussian Noise



- ◇ Our object is to design a receiver that is optimum in the sense that it minimizes the probability of making an error.
- ◇ It is convenient to subdivide the receiver into two parts—the *signal demodulator* and the *detector*.



- ◇ The function of the *signal demodulator* is to convert the received waveform $r(t)$ into an N -dimensional vector $\mathbf{r}=[r_1 r_2 \dots r_N]$ where N is the dimension of the transmitted signal waveform.
- ◇ The function of the *detector* is to decide which of the M possible signal waveforms was transmitted based on the vector \mathbf{r} .

Optimum Receiver for Signals Corrupted by Additive White Gaussian Noise



- ◇ Two realizations of the signal demodulator are described in the following section:
 - ◇ One is based on the use of *signal correlators*.
 - ◇ The second is based on the use of *matched filters*.
- ◇ The optimum detector that follows the signal demodulator is designed to minimize the probability of error.

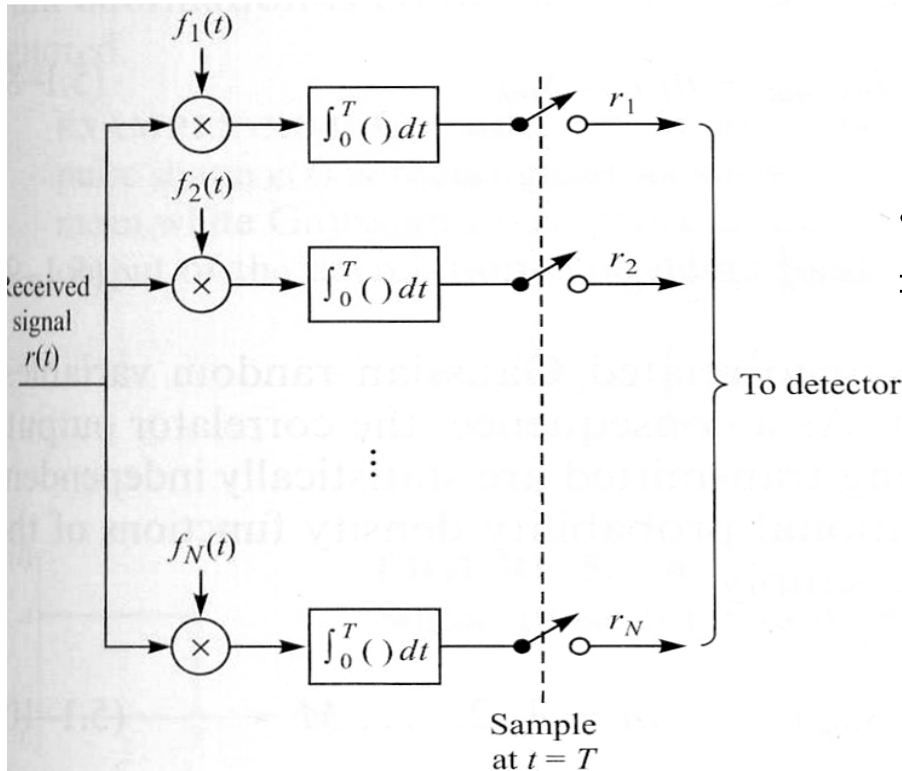
Correlation Demodulator



- ◇ We describe a *correlation demodulation* that decomposes the receiver signal and the noise into N -dimensional vectors.
- ◇ In other words, the signal and the noise are expanded into a series of linearly weighted orthonormal basis functions $\{f_n(t)\}$.
- ◇ It is assumed that the N basis function $\{f_n(t)\}$ span the signal space, so every one of the possible transmitted signals of the set $\{s_m(t)=1 \leq m \leq M\}$ can be represented as a linear combination of $\{f_n(t)\}$.
- ◇ In case of the noise, the function $\{f_n(t)\}$ do not span the noise space. However we show below that the noise terms that fall outside the signal space are irrelevant to the detection of the signal.

Correlation Demodulator

- Suppose the receiver signal $r(t)$ is passed through a parallel bank of N basis functions $\{f_n(t)\}$, as shown in the following figure:



$$\int_0^T r(t) f_k(t) dt = \int_0^T [s_m(t) + n(t)] f_k(t) dt$$

$$\Rightarrow r_k = s_{mk} + n_k, \quad k = 1, 2, \dots, N$$

$$s_{mk} = \int_0^T s_m(t) f_k(t) dt, \quad k = 1, 2, \dots, N$$

$$n_k = \int_0^T n(t) f_k(t) dt, \quad k = 1, 2, \dots, N$$

- The signal is now represented by the vector \mathbf{s}_m with components s_{mk} , $k=1, 2, \dots, N$. Their values depend on which of the M signals was transmitted.

Correlation Demodulator



- ◇ In fact, we can express the receiver signal $r(t)$ in the interval $0 \leq t \leq T$ as:

$$\begin{aligned} r(t) &= \sum_{k=1}^N s_{mk} f_k(t) + \sum_{k=1}^N n_k f_k(t) + n'(t) \\ &= \sum_{k=1}^N r_k f_k(t) + n'(t) \end{aligned}$$

- ◇ The term $n'(t)$, defined as

$$n'(t) = n(t) - \sum_{k=1}^N n_k f_k(t)$$

is a zero-mean Gaussian noise process that represents the difference between original noise process $n(t)$ and the part corresponding to the projection of $n(t)$ onto the basis functions $\{f_k(t)\}$.

Correlation Demodulator



- ◇ We shall show below that $n'(t)$ is irrelevant to the decision as to which signal was transmitted. Consequently, the decision may be based entirely on the correlator output signal and noise components $r_k = s_{mk} + n_k$, $k=1, 2, \dots, N$.
- ◇ The noise components $\{n_k\}$ are Gaussian and mean values are:

$$E(n_k) = \int_0^T E[n(t)] f_k(t) dt = 0 \quad \text{for all } n.$$

and their covariances are:

$$\begin{aligned} E(n_k n_m) &= \int_0^T \int_0^T \underbrace{E[n(t)n(\tau)]}_{\text{Autocorrelation}} f_k(t) f_m(\tau) dt d\tau \\ &= \frac{1}{2} N_0 \int_0^T \int_0^T \delta(t - \tau) f_k(t) f_m(\tau) dt d\tau \\ &= \frac{1}{2} N_0 \int_0^T f_k(t) f_m(t) dt = \frac{1}{2} N_0 \delta_{mk} \end{aligned}$$

Power spectral density is $\Phi_{nn}(f) = 1/2 N_0$ W/Hz

Conclusion: The N noise Components $\{n_k\}$ are zero-mean uncorrelated Gaussian random variables with a common variance $\sigma_n^2 = 1/2 N_0$.

Correlation Demodulator



- ◇ From the above development, it follows that the correlator output $\{r_k\}$ conditioned on the m th signal being transmitted are Gaussian random variables with mean

$$E(r_k) = E(s_{mk} + n_k) = s_{mk}$$

and equal variance

$$\sigma_r^2 = \sigma_n^2 = \frac{1}{2} N_0$$

- ◇ Since the noise components $\{n_k\}$ are uncorrelated Gaussian random variables, they are also statistically independent. As a consequence, the correlator outputs $\{r_k\}$ conditioned on the m th signal being transmitted are statistically independent Gaussian variables.

Correlation Demodulator



- ◇ The conditional probability density functions of the random variables $\mathbf{r}=[r_1 r_2 \cdots r_N]$ are:

$$p(\mathbf{r} | \mathbf{s}_m) = \prod_{k=1}^N p(r_k | s_{mk}), \quad m = 1, 2, \dots, M \quad \text{----(A)}$$

$$p(r_k | s_{mk}) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r_k - s_{mk})^2}{N_0}\right], \quad k = 1, 2, \dots, N \quad \text{----(B)}$$

By substituting Equation (B) into Equation (A), we obtain the joint conditional PDFs

$$p(\mathbf{r} | \mathbf{s}_m) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{k=1}^N \frac{(r_k - s_{mk})^2}{N_0}\right], \quad m = 1, 2, \dots, M$$

Correlation Demodulator



- ◇ The correlator outputs $(r_1 r_2 \cdots r_N)$ are *sufficient statistics* for reaching a decision on which of the M signals was transmitted, i.e., no additional relevant information can be extracted from the remaining noise process $n'(t)$.
- ◇ Indeed, $n'(t)$ is uncorrelated with the N correlator outputs $\{r_k\}$:

$$\begin{aligned}
 E[n'(t)r_k] &= E[n'(t)]s_{mk} + E[n'(t)n_k] = E[n'(t)n_k] = E\left\{\left[n(t) - \sum_{j=1}^N n_j f_j(t)\right]n_k\right\} \\
 &= \int_0^T E[n(t)n(\tau)]f_k(\tau)d\tau - \sum_{j=1}^N E(n_j n_k) f_j(t) \quad \boxed{n_k = \int_0^T n(t) f_k(t) dt} \\
 &= \int_0^T \frac{1}{2} N_0 \delta(t - \tau) f_k(\tau) d\tau - \sum_{j=1}^N \frac{1}{2} N_0 \delta_{jk} f_j(t) \\
 &= \frac{1}{2} N_0 f_k(t) - \frac{1}{2} N_0 f_k(t) = 0 \quad \text{Q.E.D.}
 \end{aligned}$$

Correlation Demodulator



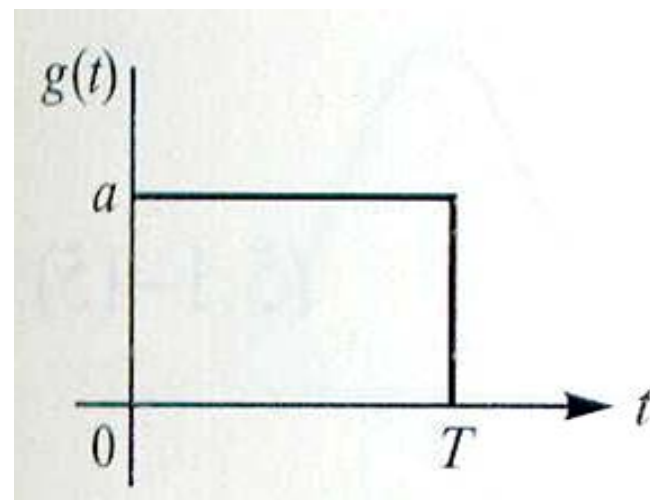
- ◇ Since $n'(t)$ and $\{r_k\}$ are Gaussian and uncorrelated, they are also statistically independent.
- ◇ Consequently, $n'(t)$ does not contain any information that is relevant to the decision as to which signal waveform was transmitted.
- ◇ All the relevant information is contained in the correlator outputs $\{r_k\}$ and, hence, $n'(t)$ can be ignored.

Correlation Demodulator



- ◇ Example.
 - ◇ Consider an M -ary baseband PAM signal set in which the basic pulse shape $g(t)$ is rectangular as shown in following figure.
 - ◇ The additive noise is a zero-mean white Gaussian noise process.
 - ◇ Let us determine the basis function $f(t)$ and the output of the correlation-type demodulator.
 - ◇ The energy in the rectangular pulse is

$$\mathcal{E}_g = \int_0^T g^2(t) dt = \int_0^T a^2 dt = a^2 T$$



Correlation Demodulator



◇ Example.(cont.)

- ◇ Since the PAM signal set has dimension $N=1$, there is only one basis function $f(t)$ given as:

$$f(t) = \frac{1}{\sqrt{a^2 T}} g(t) = \begin{cases} 1/\sqrt{T} & (0 \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases}$$

- ◇ The output of the correlation-type demodulator is:

$$r = \int_0^T r(t) f(t) dt = \frac{1}{\sqrt{T}} \int_0^T r(t) dt$$

- ◇ The correlator becomes a simple integrator when $f(t)$ is rectangular: if we substitute for $r(t)$, we obtain:

$$\begin{aligned} r &= \frac{1}{\sqrt{T}} \left\{ \int_0^T [s_m(t) + n(t)] \right\} dt = \frac{1}{\sqrt{T}} \left[\int_0^T s_m(t) dt + \int_0^T n(t) dt \right] \\ &= s_m + n \end{aligned}$$

Correlation Demodulator



◇ Example.(cont.)

- ◇ The noise term $E(n)=0$ and:

$$\begin{aligned}\sigma_n^2 &= E[n(t)n(t)] = E\left[\frac{1}{T} \int_0^T \int_0^T n(t)n(\tau) dt d\tau\right] \\ &= \frac{1}{T} \int_0^T \int_0^T E[n(t)n(\tau)] dt d\tau = \frac{N_0}{2T} \int_0^T \int_0^T \delta(t-\tau) dt d\tau \\ &= \frac{N_0}{2T} \int_0^T 1 \cdot d\tau = \frac{1}{2} N_0\end{aligned}$$

- ◇ The probability density function for the sampled output is:

$$p(r | s_m) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r - s_m)^2}{N_0}\right]$$

Matched-Filter Demodulator



- ◇ Instead of using a bank of N correlators to generate the variables $\{r_k\}$, we may use a bank of N linear filters. To be specific, let us suppose that the impulse responses of the N filters are:

$$h_k(t) = f_k(T - t), \quad 0 \leq t \leq T$$

where $\{f_k(t)\}$ are the N basis functions and $h_k(t)=0$ outside of the interval $0 \leq t \leq T$.

- ◇ The outputs of these filters are :

$$\begin{aligned} y_k(t) &= r(t) * h_k(t) \\ &= \int_0^t r(\tau) h_k(t - \tau) d\tau \\ &= \int_0^t r(\tau) f_k(T - t + \tau) d\tau, \quad k = 1, 2, \dots, N \end{aligned}$$

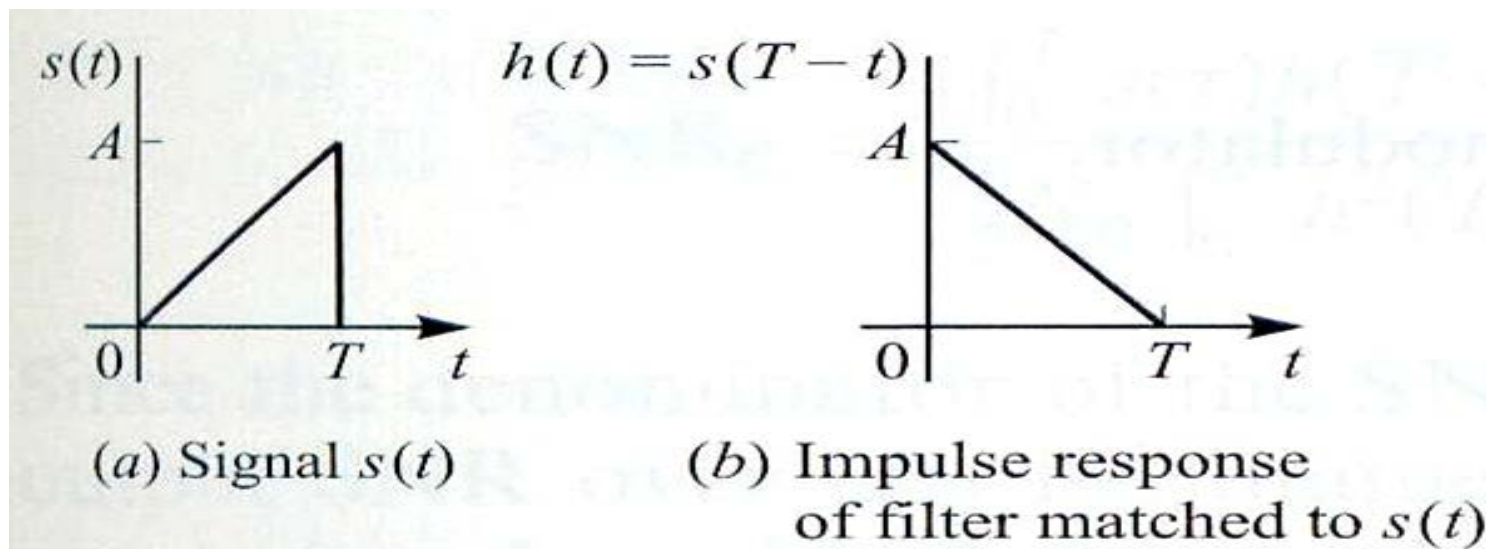
Matched-Filter Demodulator



- ◇ If we sample the outputs of the filters at $t = T$, we obtain

$$y_k(T) = \int_0^T r(\tau) f_k(\tau) d\tau = r_k, \quad k = 1, 2, \dots, N$$

- ◇ A filter whose impulse response $h(t) = s(T - t)$, where $s(t)$ is assumed to be confined to the time interval $0 \leq t \leq T$, is called *matched filter* to the signal $s(t)$.
- ◇ An example of a signal and its matched filter are shown in the following figure.



Matched-Filter Demodulator

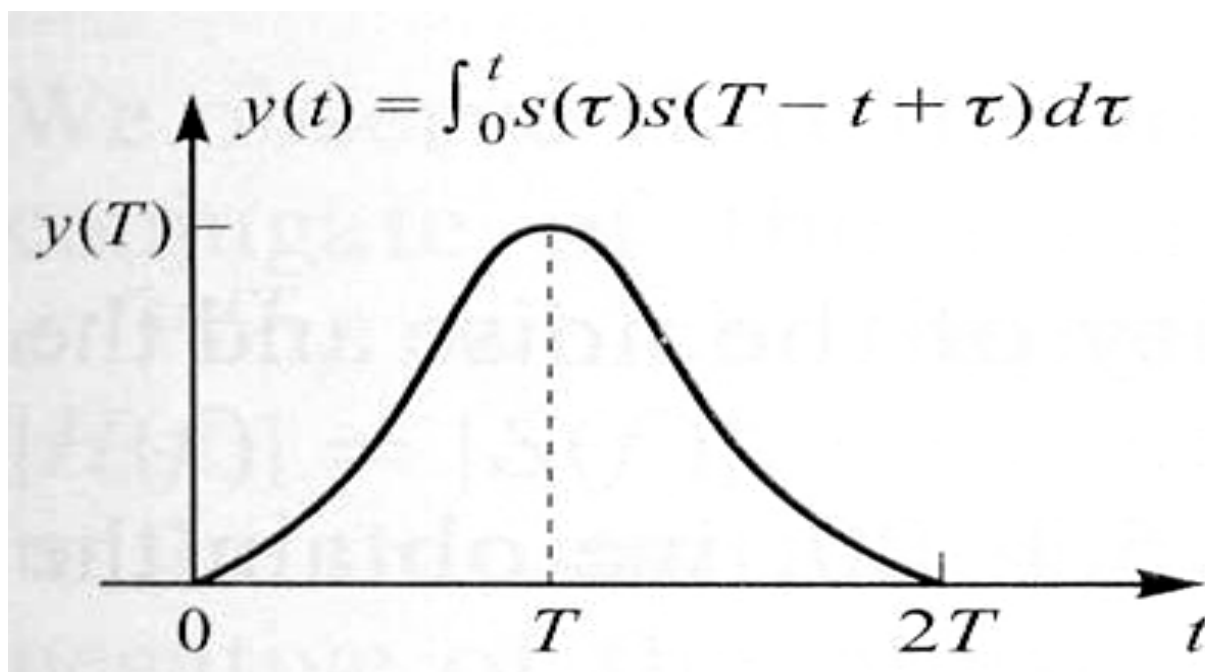


- ◇ The response of $h(t) = s(T - t)$ to the signal $s(t)$ is:

$$y(t) = s(t) * h(t) = \int_0^t s(\tau)h(t - \tau) d\tau = \int_0^t s(\tau)s(T - t + \tau) d\tau$$

which is the time-autocorrelation function of the signal $s(t)$.

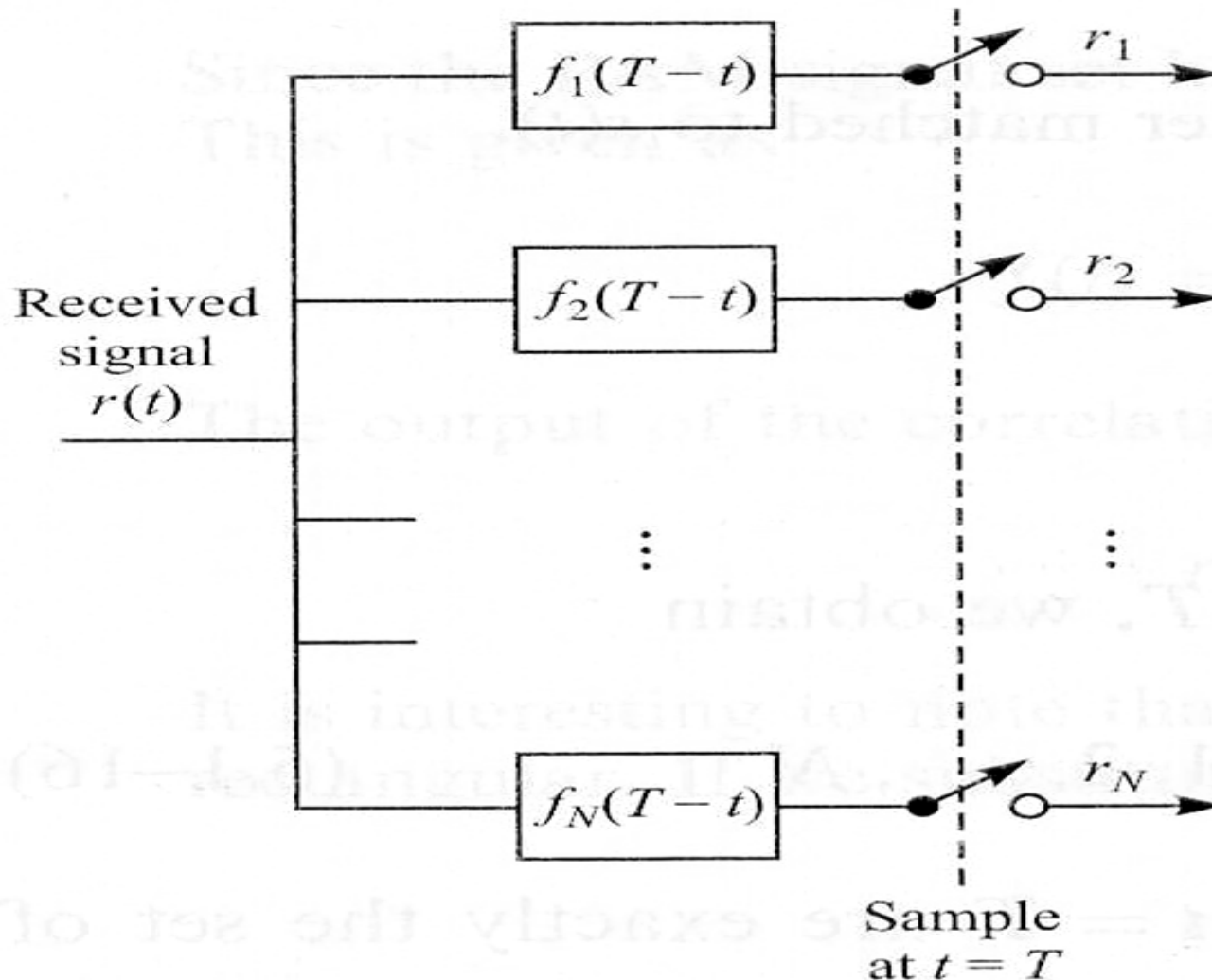
- ◇ Note that the autocorrelation function $y(t)$ is an even function of t , which attains a peak at $t=T$.



Matched-Filter Demodulator



- Matched filter demodulator that generates the observed variables $\{r_k\}$



Matched-Filter Demodulator



- ◇ Properties of the matched filter.
 - ◇ If a signal $s(t)$ is corrupted by AWGN, the filter with an impulse response matched to $s(t)$ maximizes the output signal-to-noise ratio (SNR).
 - ◇ Proof:
 - ◇ Let us assume the receiver signal $r(t)$ consists of the signal $s(t)$ and AWGN $n(t)$ which has zero-mean and $\Phi_{nn}(f) = \frac{1}{2} N_0$ W/Hz.
 - ◇ Suppose the signal $r(t)$ is passed through a filter with impulse response $h(t)$, $0 \leq t \leq T$, and its output is sampled at time $t=T$.
The output signal of the filter is:

$$\begin{aligned} y(t) &= s(t) * h(t) = \int_0^t r(\tau) h(t - \tau) d\tau \\ &= \int_0^t s(\tau) h(t - \tau) d\tau + \int_0^t n(\tau) h(t - \tau) d\tau \end{aligned}$$

Matched-Filter Demodulator



- ◇ Proof: (cont.)

- ◇ At the sampling instant $t=T$:

$$\begin{aligned}y(T) &= \int_0^T s(\tau)h(T-\tau)d\tau + \int_0^T n(\tau)h(T-\tau)d\tau \\ &= y_s(T) + y_n(T)\end{aligned}$$

- ◇ This problem is to select the filter impulse response that maximizes the output SNR_0 defined as:

$$\text{SNR}_0 = \frac{y_s^2(T)}{E[y_n^2(T)]}$$

$$\begin{aligned}E[y_n^2(T)] &= \int_0^T \int_0^T E[n(\tau)n(t)]h(T-\tau)h(T-t)dtd\tau \\ &= \frac{1}{2}N_0 \int_0^T \int_0^T \delta(t-\tau)h(T-\tau)h(T-t)dtd\tau = \frac{1}{2}N_0 \int_0^T h^2(T-t)dt\end{aligned}$$

Matched-Filter Demodulator



- ◇ Proof: (cont.)

- ◇ By substituting for $y_s(T)$ and $E[y_n^2(T)]$ into SNR_0 .

$$\tau' = T - \tau$$

$$\text{SNR}_0 = \frac{\left[\int_0^T s(\tau)h(T - \tau)d\tau \right]^2}{\frac{1}{2} N_0 \int_0^T h^2(T - t)dt} = \frac{\left[\int_0^T h(\tau')s(T - \tau')d\tau' \right]^2}{\frac{1}{2} N_0 \int_0^T h^2(T - t)dt}$$

- ◇ Denominator of the SNR depends on the energy in $h(t)$.
- ◇ The maximum output SNR over $h(t)$ is obtained by maximizing the numerator subject to the constraint that the denominator is held constant.

Matched-Filter Demodulator



- ◇ Proof: (cont.)
 - ◇ *Cauchy-Schwarz inequality*: if $g_1(t)$ and $g_2(t)$ are finite-energy signals, then

$$\left[\int_{-\infty}^{\infty} g_1(t) g_2(t) dt \right]^2 \leq \int_{-\infty}^{\infty} g_1^2(t) dt \int_{-\infty}^{\infty} g_2^2(t) dt$$

with equality when $g_1(t) = Cg_2(t)$ for any arbitrary constant C .

- ◇ If we set $g_1(t) = h_1(t)$ and $g_2(t) = s(T-t)$, it is clear that the SNR is maximized when $h(t) = Cs(T-t)$.

Matched-Filter Demodulator



- ◇ Proof: (cont.)
 - ◇ The output (maximum) SNR obtained with the matched filter is:

$$\begin{aligned} \text{SNR}_0 &= \frac{\left[\int_0^T s(\tau) h(T - \tau) d\tau \right]^2}{\frac{1}{2} N_0 \int_0^T h^2(T - t) dt} = \frac{\left[\int_0^T s(\tau) C s(T - (T - \tau)) d\tau \right]^2}{\frac{1}{2} N_0 \int_0^T C^2 s^2(T - (T - t)) dt} \\ &= \frac{2}{N_0} \int_0^T s^2(t) dt = \frac{2\mathcal{E}}{N_0} \end{aligned}$$

- ◇ Note that the output SNR from the matched filter depends on the energy of the waveform $s(t)$ but not on the detailed characteristics of $s(t)$.

Matched-Filter Demodulator



- ◇ Frequency-domain interpretation of the matched filter
 - ◇ Since $h(t)=s(T - t)$, the *Fourier transform* of this relationship is:

$$\begin{aligned} H(f) &= \int_0^T s(T - t)e^{-j2\pi ft} dt \quad \text{let } \tau = T - t \\ &= \left[\int_0^T s(\tau)e^{j2\pi f\tau} d\tau \right] e^{-j2\pi fT} = S^*(f)e^{-j2\pi fT} \end{aligned}$$

- ◇ The matched filter has a frequency response that is the complex conjugate of the transmitted signal spectrum multiplied by the phase factor $e^{-j2\pi fT}$ (sampling delay of T).
- ◇ In other words, $|H(f)|=|S(f)|$, so that the magnitude response of the matched filter is identical to the transmitted signal spectrum.
- ◇ On the other hand, the phase of $H(f)$ is the negative of the phase of $S(f)$.

Matched-Filter Demodulator



- ◇ Frequency-domain interpretation of the matched filter

- ◇ If the signal $s(t)$ with spectrum $S(f)$ is passed through the matched filter, the filter output has a spectrum

$$Y(f) = |S(f)|^2 e^{-j2\pi fT}. \quad S(f) \cdot S^*(f) e^{-j2\pi fT}$$

- ◇ The output waveform is:

$$y_s(t) = \int_{-\infty}^{\infty} Y(f) e^{j2\pi ft} df = \int_{-\infty}^{\infty} |S(f)|^2 e^{-j2\pi fT} e^{j2\pi ft} df$$

- ◇ By sampling the output of the matched filter at $t = T$, we obtain (from Parseval's relation):

$$y_s(T) = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_0^T s^2(t) dt = \mathcal{E}$$

- ◇ The noise at the output of the matched filter has a power spectral density (See Chapter 2, Page 109)

$$\Phi_0(f) = \frac{1}{2} |H(f)|^2 N_0$$

Matched-Filter Demodulator



- ◇ Frequency-domain interpretation of the matched filter
 - ◇ The total noise power at the output of the matched filter is

$$\begin{aligned} P_n &= \int_{-\infty}^{\infty} \Phi_0(f) df \\ &= \frac{1}{2} N_0 \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{1}{2} N_0 \int_{-\infty}^{\infty} |S(f)|^2 df = \frac{1}{2} \mathcal{E} N_0 \end{aligned}$$

- ◇ The output SNR is simply the ratio of the signal power P_s , given by $P_s = y_s^2(T)$, to the noise power P_n .

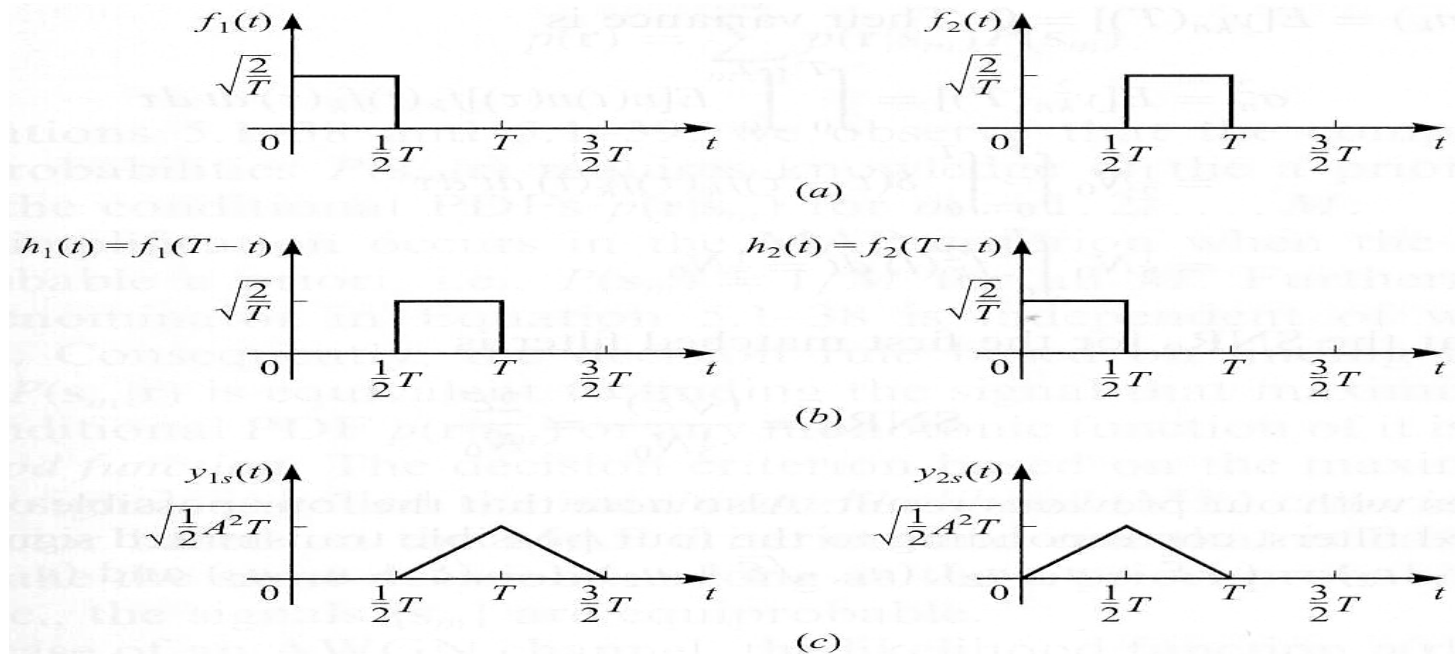
$$\text{SNR}_0 = \frac{P_s}{P_n} = \frac{\mathcal{E}^2}{\frac{1}{2} \mathcal{E} N_0} = \frac{2\mathcal{E}}{N_0}$$

Matched-Filter Demodulator



◇ Example:

- ◇ $M=4$ biorthogonal signals are constructed from the two orthogonal signals shown in the following figure for transmitting information over an AWGN channel. The noise is assumed to have a zero-mean and power spectral density $\frac{1}{2} N_0$.



Matched-Filter Demodulator



◇ Example: (cont.)

- ◇ The $M=4$ biorthogonal signals have dimensions $N=2$ (shown in figure a):

$$f_1(t) = \begin{cases} \sqrt{2/T} & (0 \leq t \leq \frac{1}{2}T) \\ 0 & (\text{otherwise}) \end{cases}$$
$$f_2(t) = \begin{cases} \sqrt{2/T} & (\frac{1}{2}T \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases}$$

- ◇ The impulse responses of the two matched filters are (figure b):

$$h_1(t) = f_1(T-t) = \begin{cases} \sqrt{2/T} & (\frac{1}{2}T \leq t \leq T) \\ 0 & (\text{otherwise}) \end{cases}$$
$$h_2(t) = f_2(T-t) = \begin{cases} \sqrt{2/T} & (0 \leq t \leq \frac{1}{2}T) \\ 0 & (\text{otherwise}) \end{cases}$$

Matched-Filter Demodulator



◇ Example: (cont.)

◇ If $s_1(t)$ is transmitted, the responses of the two matched filters are as shown in figure c, where the signal amplitude is A .

◇ Since $y_1(t)$ and $y_2(t)$ are sampled at $t=T$, we observe that

$$y_{1S}(T) = \sqrt{\frac{1}{2} A^2 T} \quad , \quad y_{2S}(T) = 0.$$

◇ From equation Page 28, we have $\frac{1}{2} A^2 T = \mathcal{E}$ and the received vector is:

$$\mathbf{r} = [r_1 \quad r_2] = \begin{bmatrix} \sqrt{\mathcal{E}} + n_1 & n_2 \end{bmatrix}$$

where $n_1(t) = y_{1n}(T)$ and $n_2 = y_{2n}(T)$ are the noise components at the outputs of the matched filters, given by

$$y_{kn}(T) = \int_0^T n(t) f_k(t) dt, \quad k = 1, 2$$

Matched-Filter Demodulator



◇ Example: (cont.)

- ◇ Clearly, $E(n_k) = E[y_{kn}(T)]$ and their variance is

$$\begin{aligned}\sigma_n^2 &= E[y_{kn}^2(T)] = \int_0^T \int_0^T E[n(t)n(\tau)] f_k(t) f_k(\tau) dt d\tau \\ &= \frac{1}{2} N_0 \int_0^T \int_0^T \delta(t - \tau) f_k(\tau) f_k(t) dt d\tau \\ &= \frac{1}{2} N_0 \int_0^T f_k^2(t) dt = \frac{1}{2} N_0\end{aligned}$$

- ◇ Observe that the SNR_0 for the first matched filter is

$$\text{SNR}_0 = \frac{(\sqrt{\mathcal{E}})^2}{\frac{1}{2} N_0} = \frac{2\mathcal{E}}{N_0}$$

Matched-Filter Demodulator



- ◇ Example: (cont.)
 - ◇ This result agrees with our previous result.
 - ◇ We can also note that the four possible outputs of the two matched filters, corresponding to the four possible transmitted signals are:

$$(r_1, r_2) = (\sqrt{\varepsilon} + n_1, n_2), (n_1, \sqrt{\varepsilon} + n_2), (-\sqrt{\varepsilon} + n_1, n_2) \text{ and } (n_1, -\sqrt{\varepsilon} + n_2)$$

The Optimum Detector



- ◇ Our goal is to design a signal detector that makes a decision on the transmitted signal in each signal interval based on the observation of the vector \mathbf{r} in each interval such that the probability of a correct decision is maximized.
- ◇ We assume that there is no memory in signals transmitted in successive signal intervals.
- ◇ We consider a decision rule based on the computation of the *posterior probabilities* defined as

$$P(\mathbf{s}_m|\mathbf{r}) \equiv P(\text{signal } \mathbf{s}_m \text{ was transmitted}|\mathbf{r}), \quad m=1,2,\dots,M.$$

- ◇ The decision criterion is based on selecting the signal corresponding to the maximum of the set of posterior probabilities $\{ P(\mathbf{s}_m|\mathbf{r}) \}$. This decision criterion is called the *maximum a posterior probability (MAP) criterion*.

The Optimum Detector



- Using *Bayes' rule*, the posterior probabilities may be expressed as

$$P(\mathbf{s}_m | \mathbf{r}) = \frac{p(\mathbf{r} | \mathbf{s}_m)P(\mathbf{s}_m)}{p(\mathbf{r})} \quad \text{---(A)}$$

where $P(\mathbf{s}_m)$ is the *a priori probability* of the m th signal being transmitted.

- The denominator of (A), which is independent of which signal is transmitted, may be expressed as

$$p(\mathbf{r}) = \sum_{m=1}^M p(\mathbf{r} | \mathbf{s}_m)P(\mathbf{s}_m)$$

- Some simplification occurs in the MAP criterion when the M signal are equally probable a priori, i.e., $P(\mathbf{s}_m)=1/M$.
- The decision rule based on finding the signal that maximizes $P(\mathbf{s}_m|\mathbf{r})$ is equivalent to finding the signal that maximizes $P(\mathbf{r}|\mathbf{s}_m)$.

The Optimum Detector



- ◇ The conditional PDF $P(\mathbf{r}|\mathbf{s}_m)$ or any monotonic function of it is usually called the *likelihood function*.
- ◇ The decision criterion based on the maximum of $P(\mathbf{r}|\mathbf{s}_m)$ over the M signals is called *maximum-likelihood (ML) criterion*.
- ◇ We observe that a detector based on the MAP criterion and one that is based on the ML criterion make the same decisions as long as a priori probabilities $P(\mathbf{s}_m)$ are all equal.
- ◇ In the case of an AWGN channel, the likelihood function $p(\mathbf{r}|\mathbf{s}_m)$ is given by:

$$p(\mathbf{r}|\mathbf{s}_m) = \frac{1}{(\pi N_0)^{N/2}} \exp \left[-\sum_{k=1}^N \frac{(r_k - s_{mk})^2}{N_0} \right], \quad m = 1, 2, \dots, M$$

$$\ln p(\mathbf{r} | \mathbf{s}_m) = \underbrace{-\frac{1}{2} N \ln(\pi N_0)}_{\text{constant}} - \frac{1}{N_0} \sum_{k=1}^N (r_k - s_{mk})^2$$

The Optimum Detector



- ◇ The maximum of $\ln p(\mathbf{r}|\mathbf{s}_m)$ over \mathbf{s}_m is equivalent to finding the signal \mathbf{s}_m that minimizes the *Euclidean distance*:

$$D(\mathbf{r}, \mathbf{s}_m) = \sum_{k=1}^N (r_k - s_{mk})^2$$

- ◇ We called $D(\mathbf{r}, \mathbf{s}_m)$, $m=1, 2, \dots, M$, the *distance metrics*.
- ◇ Hence, for the AWGN channel, the decision rule based on the ML criterion reduces to finding the signal \mathbf{s}_m that is closest in distance to the receiver signal vector \mathbf{r} .
- ◇ We shall refer to this decision rule as *minimum distance detection*.

The Optimum Detector



- ◇ Expanding the distance metrics:

$$\begin{aligned} D(\mathbf{r}, \mathbf{s}_m) &= \sum_{n=1}^N r_n^2 - 2 \sum_{n=1}^N r_n s_{mn} + \sum_{n=1}^N s_{mn}^2 \\ &= \|\mathbf{r}\|^2 - 2\mathbf{r} \cdot \mathbf{s}_m + \|\mathbf{s}_m\|^2, \quad m = 1, 2, \dots, M \end{aligned}$$

- ◇ The term $\|\mathbf{r}\|^2$ is common to all distance metrics, and, hence, it may be ignored in the computations of the metrics.
- ◇ The result is a set of *modified distance metrics*.

$$D'(\mathbf{r}, \mathbf{s}_m) = -2\mathbf{r} \cdot \mathbf{s}_m + \|\mathbf{s}_m\|^2$$

- ◇ Note that selecting the signal \mathbf{s}_m that minimizes $D'(\mathbf{r}, \mathbf{s}_m)$ is equivalent to selecting the signal that maximizes the metrics $C(\mathbf{r}, \mathbf{s}_m) = -D'(\mathbf{r}, \mathbf{s}_m)$,

$$C(\mathbf{r}, \mathbf{s}_m) = 2\mathbf{r} \cdot \mathbf{s}_m - \|\mathbf{s}_m\|^2$$

The Optimum Detector



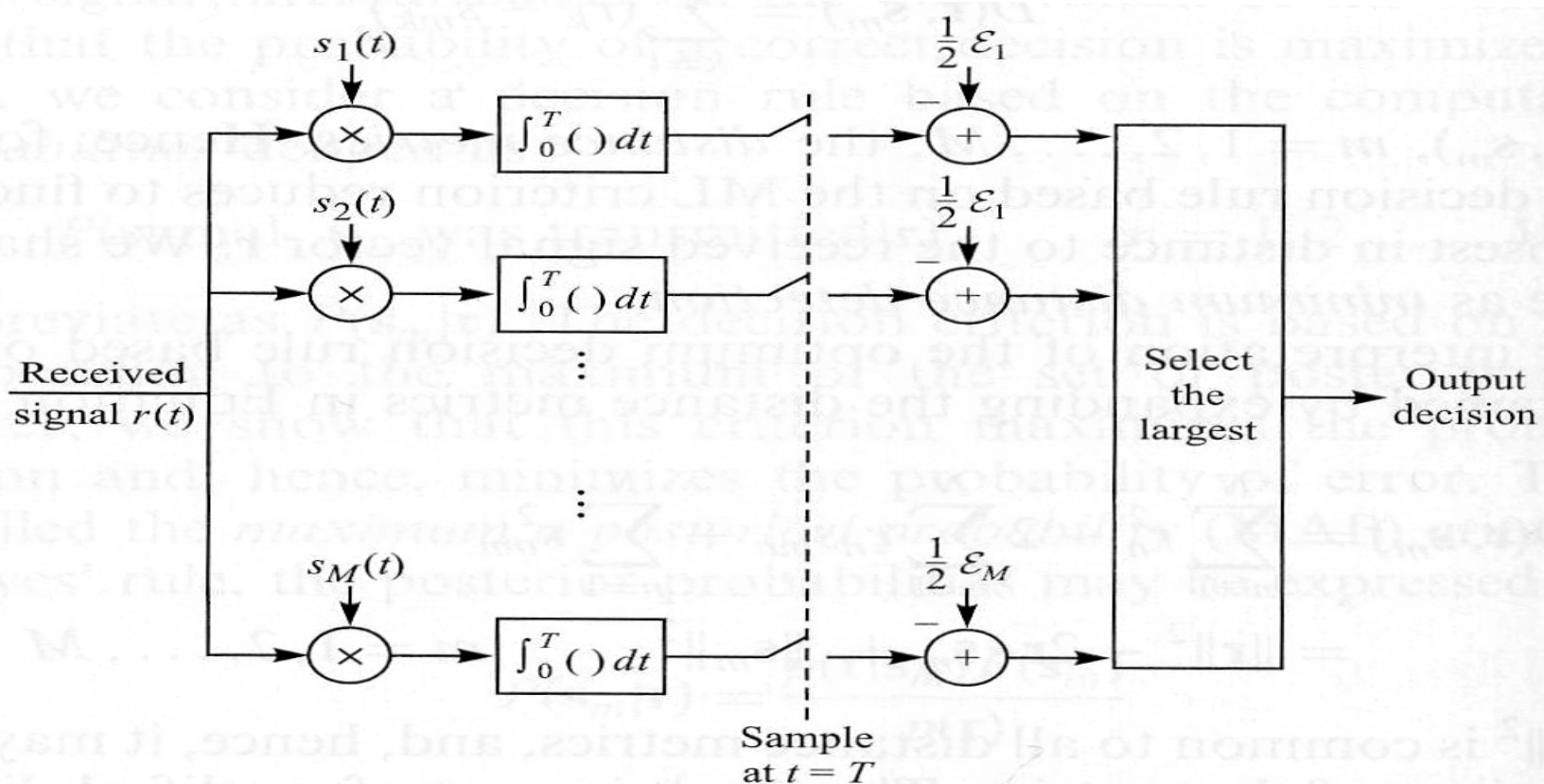
- ◇ The term $\mathbf{r} \cdot \mathbf{s}_m$ represents the projection of the signal vector onto each of the M possible transmitted signal vectors.
- ◇ The value of each of these projection is a measure of the correlation between the receiver vector and the m th signal. For this reason, we call $C(\mathbf{r}, \mathbf{s}_m)$, $m=1,2,\dots,M$, the *correlation metrics* for deciding which of the M signals was transmitted.
- ◇ Finally, the terms $\|\mathbf{s}_m\|^2 = \mathcal{E}_m$, $m=1,2,\dots,M$, may be viewed as bias terms that serve as compensation for signal sets that have unequal energies.
- ◇ If all signals have the same energy, $\|\mathbf{s}_m\|^2$ may also be ignored.
- ◇ Correlation metrics can be expressed as:

$$C(\mathbf{r}, \mathbf{s}_m) = 2 \int_0^T r(t) s_m(t) dt - \mathcal{E}_m, \quad m = 0, 1, \dots, M$$

The Optimum Detector



- These metrics can be generated by a demodulator that cross-correlates the received signal $r(t)$ with each of the M possible transmitted signals and adjusts each correlator output for the bias in the case of unequal signal energies.



The Optimum Detector



- ◇ We have demonstrated that the optimum ML detector computes a set of M distances $D(\mathbf{r}, \mathbf{s}_m)$ or $D'(\mathbf{r}, \mathbf{s}_m)$ and selects the signal corresponding to the smallest (distance) metric.
- ◇ Equivalently, the optimum ML detector computes a set of M correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ and selects the signal corresponding to the largest correlation metric.
- ◇ The above development for the optimum detector treated the important case in which all signals are equal probable. In this case, the MAP criterion is equivalent to the ML criterion.
- ◇ When the signals are not equally probable, the optimum MAP detector bases its decision on the probabilities given by:

$$P(\mathbf{s}_m | \mathbf{r}) = \frac{p(\mathbf{r} | \mathbf{s}_m)P(\mathbf{s}_m)}{p(\mathbf{r})} \quad \text{or} \quad PM(\mathbf{r}, \mathbf{s}_m) = p(\mathbf{r} | \mathbf{s}_m)P(\mathbf{s}_m)$$

The Optimum Detector



◇ Example:

- ◇ Consider the case of binary PAM signals in which the two possible signal points are $s_1 = -s_2 = \sqrt{\varepsilon_b}$, where ε_b is the energy per bit. The priori probabilities are $P(s_1)=p$ and $P(s_2)=1-p$. Let us determine the metrics for the optimum MAP detector when the transmitted signal is corrupted with AWGN.
- ◇ The receiver signal vector for binary PAM is:

$$r = \pm\sqrt{\mathcal{E}_b} + y_n(T)$$

where $y_n(T)$ is a zero mean Gaussian random variable with variance $\sigma_n^2 = \frac{1}{2}N_0$.

The Optimum Detector



◇ Example: (cont.)

- ◇ The conditional PDF $P(\mathbf{r}|\mathbf{s}_m)$ for two signals are

$$p(r | s_1) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r - \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right]$$

$$p(r | s_2) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r + \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right]$$

- ◇ Then the metrics $PM(\mathbf{r}, \mathbf{s}_1)$ and $PM(\mathbf{r}, \mathbf{s}_2)$ are

$$PM(\mathbf{r}, \mathbf{s}_1) = p \cdot p(r | s_1) = \frac{p}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r - \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right]$$

$$PM(\mathbf{r}, \mathbf{s}_2) = (1 - p) \cdot p(r | s_2) = \frac{1 - p}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r + \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2}\right]$$

The Optimum Detector



◇ Example: (cont.)

- ◇ If $PM(\mathbf{r}, \mathbf{s}_1) > PM(\mathbf{r}, \mathbf{s}_2)$, we select s_1 as the transmitted signal: otherwise, we select s_2 . This decision rule may be expressed as:

$$\frac{PM(\mathbf{r}, \mathbf{s}_1)}{PM(\mathbf{r}, \mathbf{s}_2)} \underset{s_2}{\overset{s_1}{>}} 1$$

$$\frac{PM(\mathbf{r}, \mathbf{s}_1)}{PM(\mathbf{r}, \mathbf{s}_2)} = \frac{p}{1-p} \exp \left[\frac{(r + \sqrt{\mathcal{E}_b})^2 - (r - \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2} \right]$$

$$\frac{(r + \sqrt{\mathcal{E}_b})^2 - (r - \sqrt{\mathcal{E}_b})^2}{2\sigma_n^2} \underset{s_2}{\overset{s_1}{>}} \ln \frac{1-p}{p}$$

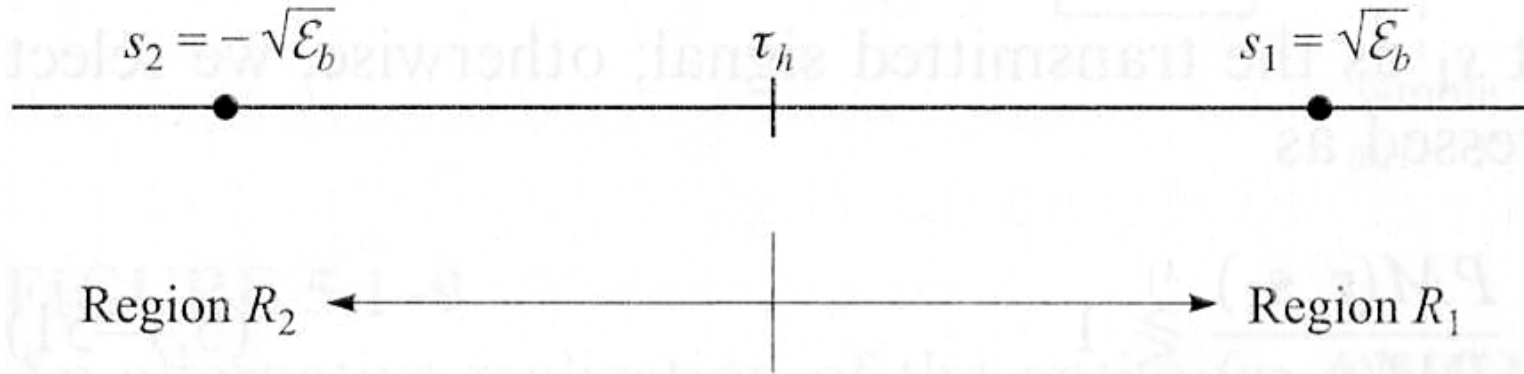
$$\sqrt{\mathcal{E}_b} r \underset{s_2}{\overset{s_1}{>}} \frac{1}{2} \sigma_n^2 \ln \frac{1-p}{p} = \frac{1}{4} N_0 \ln \frac{1-p}{p}$$

The Optimum Detector



◇ Example: (cont.)

- ◇ The threshold is $\frac{1}{4} N_0 \ln \frac{1-p}{p}$, denoted by τ_h , divides the real line into two regions, say R_1 and R_2 , where R_1 consists of the set of points that are greater than τ_h and R_2 consists of the set of points that are less than τ_h .
- ◇ If $r\sqrt{\mathcal{E}_b} > \tau_h$, the decision is made that s_1 was transmitted.
- ◇ If $r\sqrt{\mathcal{E}_b} < \tau_h$, the decision is made that s_2 was transmitted.



The Optimum Detector



- ◇ Example: (cont.)
 - ◇ The threshold τ_h depends on N_0 and p . If $p=1/2$, $\tau_h=0$.
 - ◇ If $p>1/2$, the signal point s_1 is more probable and, hence, $\tau_h<0$. In this case, the region R_1 is larger than R_2 , so that s_1 is more likely to be selected than s_2 .
 - ◇ The average probability of error is minimized
 - ◇ It is interesting to note that in the case of unequal priori probabilities, it is necessary to know not only the values of the priori probabilities but also the value of the power spectral density N_0 , or equivalently, the noise-to-signal ratio, in order to compute the threshold.
 - ◇ When $p=1/2$, the threshold is zero, and knowledge of N_0 is not required by the detector.

The Optimum Detector



- ◇ Proof of “the decision rule based on the maximum-likelihood criterion minimizes the probability of error when the M signals are equally probable a priori”.
- ◇ Let us denote by R_m the region in the N -dimensional space for which we decide that signal $s_m(t)$ was transmitted when the vector $\mathbf{r}=[r_1 r_2 \cdots r_N]$ is received.
- ◇ The probability of a correct decision given that $s_m(t)$ was transmitted is:
$$P(c | s_m) \cdot p(s_m) = \int_{R_m} p(\mathbf{r} | s_m) \cdot p(s_m) d\mathbf{r}$$
- ◇ The average probability of a correct decision is: $\frac{1}{M}$
$$P(c) = \sum_{m=1}^M \frac{1}{M} P(c | s_m) = \sum_{m=1}^M \frac{1}{M} \int_{R_m} p(\mathbf{r} | s_m) d\mathbf{r}$$
- ◇ Note that $P(c)$ is maximized by selecting the signal s_m if $p(\mathbf{r}|s_m)$ is larger than $p(\mathbf{r}|s_k)$ for all $m \neq k$. Q.E.D.

The Optimum Detector



- ◇ Similarly for the MAP criterion, when the M signals are not equally probable, the average probability of a correct decision is

$$P(c) = \sum_{m=1}^M \int_{R_m} P(\mathbf{s}_m | \mathbf{r}) p(\mathbf{r}) d\mathbf{r}$$

same for all \mathbf{s}_m .

- ◇ In order for $P(c)$ to be as large as possible, the points that are to be included in each particular region R_m are those for which $P(\mathbf{s}_m | \mathbf{r})$ exceeds all the other posterior probabilities. Q.E.D.
- ◇ We conclude that MAP criterion maximize the probability of correct detection.

Performance of the Optimum Receiver for Memoryless Modulation



- ◇ Probability of Error for Binary Modulation
- ◇ Probability of Error for M -ary Orthogonal Signals
- ◇ Probability of Error for M -ary Biorthogonal Signals
- ◇ Probability of Error for M -ary PAM
- ◇ Probability of Error for M -ary PSK
- ◇ Differential PSK (DPSK) and Its Performance
- ◇ Probability of Error for QAM
- ◇ Comparison of Digital Modulation Methods

Probability of Error for Binary Modulation



- ◇ Let us consider binary PAM signals where the two signal waveforms are $s_1(t)=g(t)$ and $s_2(t)=-g(t)$, and $g(t)$ is an arbitrary pulse that is nonzero in the interval $0 \leq t \leq T_b$ and zero elsewhere.
- ◇ Since $s_1(t) = -s_2(t)$, these signals are said to be *antipodal*.
- ◇ The energy in the pulse $g(t)$ is ϵ_b .
- ◇ PAM signals are one-dimensional, and, their geometric representation is simply the one-dimensional vector:

$$s_1 = \sqrt{\epsilon_b} \quad , \quad s_2 = -\sqrt{\epsilon_b}$$

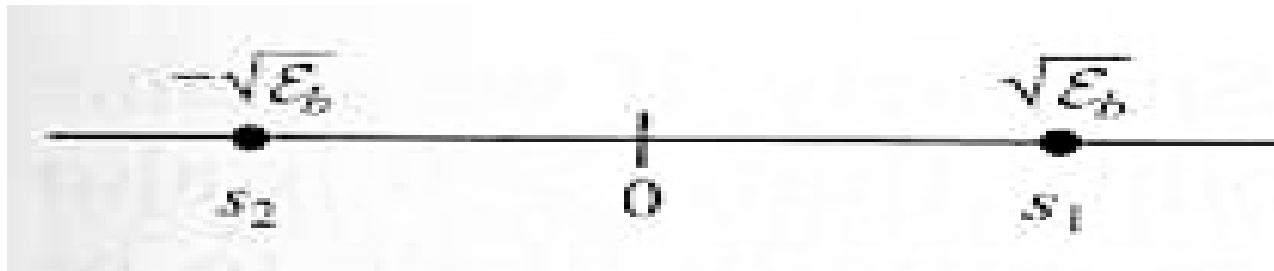


Figure (A)

Probability of Error for Binary Modulation



- ◇ Let us assume that the two signals are equally likely and that signal $s_1(t)$ was transmitted. Then, the received signal from the (matched filter or correlation) demodulator is

$$r = s_1 + n = \sqrt{\mathcal{E}_b} + n$$

where n represents the additive Gaussian noise component, which has zero mean and variance $\sigma_n^2 = \frac{1}{2}N_0$.

- ◇ In this case, the decision rule based on the correlation metric given by $C(\mathbf{r}, \mathbf{s}_m) = 2\mathbf{r} \cdot \mathbf{s}_m - \|\mathbf{s}_m\|^2$ (Page 39) compares r with the threshold zero. If $r > 0$, the decision is made in favor of $s_1(t)$, and if $r < 0$, the decision is made that $s_2(t)$ was transmitted.

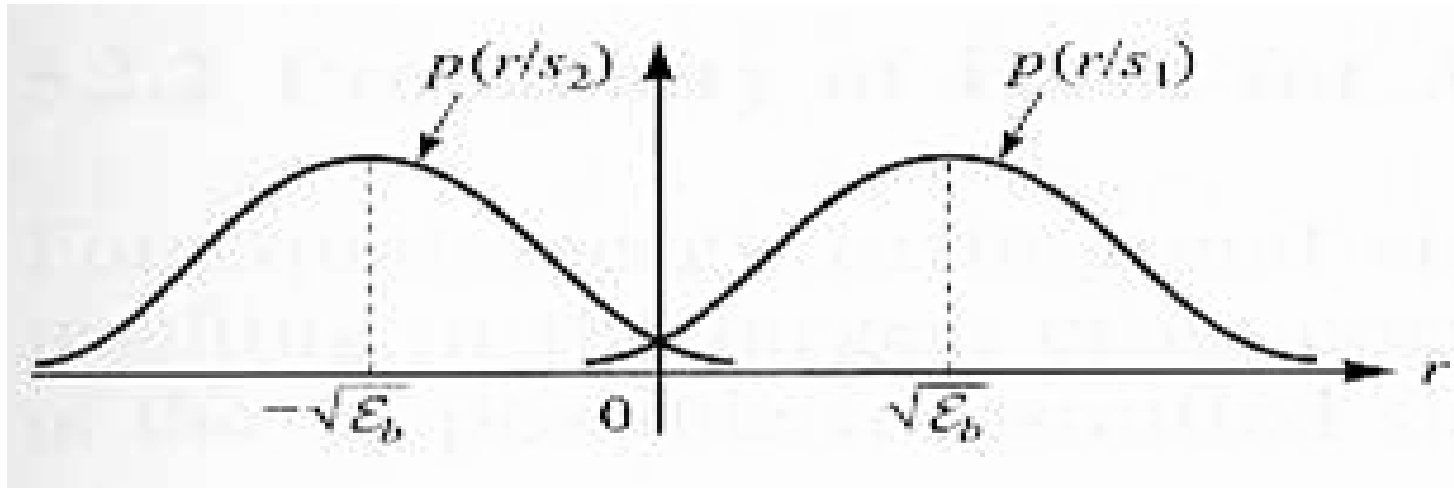
Probability of Error for Binary Modulation



- ◇ The two conditional PDFs of r are:

$$p(r | s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-(r - \sqrt{\varepsilon_b})^2 / N_0}$$

$$p(r | s_2) = \frac{1}{\sqrt{\pi N_0}} e^{-(r + \sqrt{\varepsilon_b})^2 / N_0}$$



Probability of Error for Binary Modulation



- Given that $s_1(t)$ was transmitted, the probability of error is simply the probability that $r < 0$.

$$P(e | s_1) = \int_{-\infty}^0 p(r | s_1) dr = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^0 \exp \left[-\frac{(r - \sqrt{\mathcal{E}_b})^2}{N_0} \right] dr$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{2\mathcal{E}_b}/N_0} e^{-x^2/2} dx \quad x = \frac{(r - \sqrt{\mathcal{E}_b})}{\sqrt{N_0/2}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2\mathcal{E}_b}/N_0}^{\infty} e^{-x^2/2} dx \quad (x = -x)$$

$$= Q \left(\sqrt{\frac{2\mathcal{E}_b}{N_0}} \right)$$

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2} dx \quad t \geq 0$$

Probability of Error for Binary Modulation



- ◇ If we assume that $s_2(t)$ was transmitted, $r = -\sqrt{\varepsilon_b} + n$ and the probability that $r > 0$ is also $P(e | s_2) = Q\left(\sqrt{2\varepsilon_b / N_0}\right)$. Since the signal $s_1(t)$ and $s_2(t)$ are equally likely to be transmitted, the average probability of error is

$$P_b = \frac{1}{2} P(e | s_1) + \frac{1}{2} P(e | s_2) = Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right) \quad \text{--(A)}$$

- ◇ Two important characteristics of this performance measure:
 - ◇ First, we note that the probability of error depends only on the ratio ε_b / N_0 .
 - ◇ Secondly, we note that $2\varepsilon_b / N_0$ is also the output SNR_0 from the matched-filter (and correlation) demodulator.
- ◇ The ratio ε_b / N_0 is usually called the *signal-to-noise ratio per bit*.

Probability of Error for Binary Modulation



- ◇ We also observe that the probability of error may be expressed in terms of the distance between the two signals s_1 and s_2 .
- ◇ From Page 52, we observe that the two signals are separated by the distance $d_{12} = 2\sqrt{\varepsilon_b}$. By substituting $\varepsilon_b = \frac{1}{4}d_{12}^2$ into Equation (A), we obtain

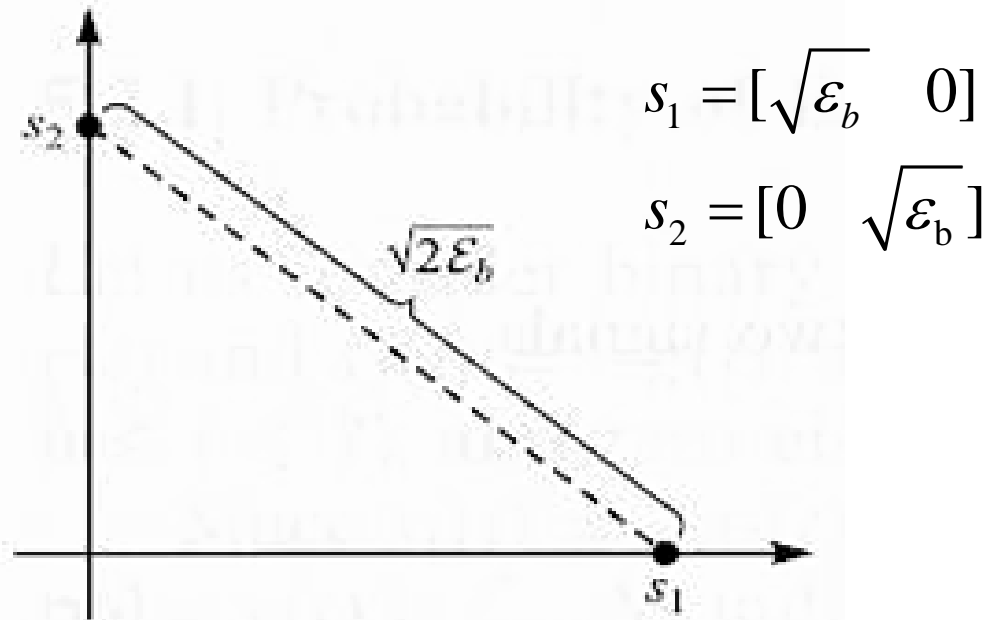
$$P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

- ◇ This expression illustrates the dependence of the error probability on the distance between the two signals points.

Probability of Error for Binary Modulation



- ◇ Error probability for binary orthogonal signals
 - ◇ The signal vectors s_1 and s_2 are two-dimensional.



where ϵ_b denote the energy for each of the waveforms. Note that the distance between these signal points is $d_{12} = \sqrt{2\epsilon_b}$.

Probability of Error for Binary Modulation



- ◇ Error probability for binary orthogonal signals
 - ◇ To evaluate the probability of error, let us assume that s_1 was transmitted. Then, the received vector at the output of the demodulator is $r = [\sqrt{\varepsilon_b} + n_1 \quad n_2]$.
 - ◇ We can now substitute for \mathbf{r} into the correlation metrics given by $C(\mathbf{r}, \mathbf{s}_m) = 2\mathbf{r} \cdot \mathbf{s}_m - \|\mathbf{s}_m\|^2$ to obtain $C(\mathbf{r}, \mathbf{s}_1)$ and $C(\mathbf{r}, \mathbf{s}_2)$.
 - ◇ The probability of error is the probability that $C(\mathbf{r}, \mathbf{s}_2) > C(\mathbf{r}, \mathbf{s}_1)$.

$$\Rightarrow (2n_2\sqrt{\varepsilon_b} - \varepsilon_b) > (2\varepsilon_b + 2n_1\sqrt{\varepsilon_b} - \varepsilon_b)$$

$$n_2 > n_1 + \sqrt{\varepsilon_b}$$

$$P(e | s_1) = P[C(r, s_2) > C(r, s_1)] = P[n_2 - n_1 > \sqrt{\varepsilon_b}]$$

Probability of Error for Binary Modulation



- ◇ Error probability for binary orthogonal signals
 - ◇ Since n_1 and n_2 are zero-mean statistically independent Gaussian random variables each with variance $\frac{1}{2} N_0$, the random variable $x = n_2 - n_1$ is zero-mean Gaussian with variance N_0 . Hence,

$$\begin{aligned} P(n_2 - n_1 > \sqrt{\varepsilon_b}) &= \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{\varepsilon_b}}^{\infty} e^{-x^2/2N_0} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\varepsilon_b/N_0}}^{\infty} e^{-x^2/2} dx = Q\left(\sqrt{\frac{\varepsilon_b}{N_0}}\right) \end{aligned}$$

- ◇ The same error probability is obtained when we assume that \mathbf{s}_2 is transmitted:

$$P_b = Q\left(\sqrt{\frac{\varepsilon_b}{N_0}}\right) = Q(\sqrt{\gamma_b}) \quad \text{where } \gamma_b \text{ is the SNR per bit.}$$

Probability of Error for Binary Modulation



- ◇ If we compare the probability of error for binary antipodal signals with that for binary orthogonal signals, we find that orthogonal signals required a factor of 2 increase in energy to achieve the same error probability as antipodal signals.
- ◇ Since $10 \log_{10} 2 = 3 \text{ dB}$, we say that orthogonal signals are 3 dB poorer than antipodal signals. The difference of 3 dB is simply due to the distance between the two signal points, which is $d_{12}^2 = 2\varepsilon_b$ for orthogonal signals, whereas $d_{12}^2 = 4\varepsilon_b$ for antipodal signals.
- ◇ The error probability versus $10 \log_{10} \varepsilon_b/N_0$ for these two types of signals is shown in the following figure (B). As observed from this figure, at any given error probability, the ε_b/N_0 required for orthogonal signals is 3 dB more than that for antipodal signals.

Probability of Error for Binary Modulation



- ◇ Probability of error for binary signals

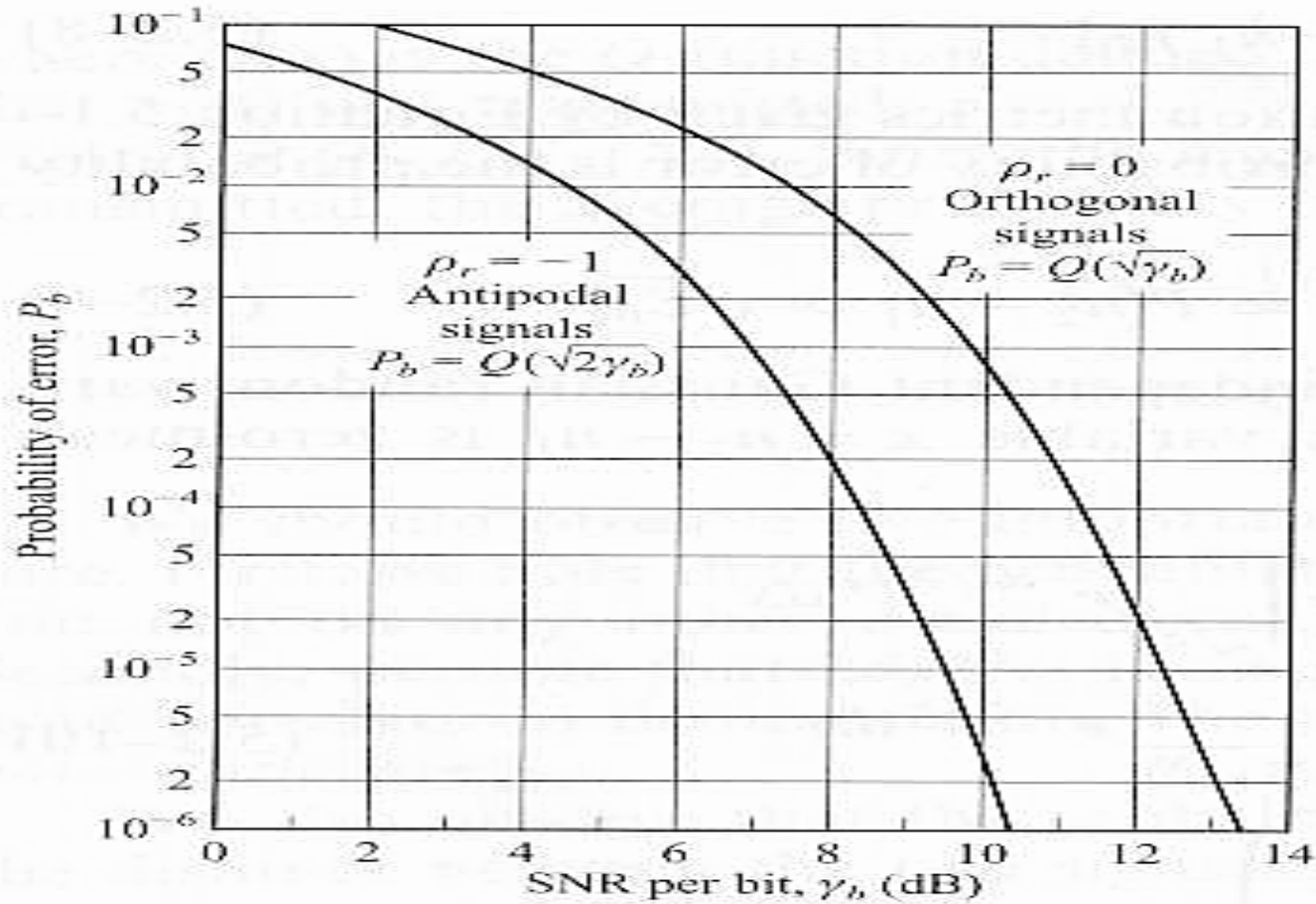


Figure (B)

Probability of Error for M-ary Orthogonal Signals



- ◇ For equal-energy orthogonal signals, the optimum detector selects the signal resulting in the largest cross correlation between the received vector \mathbf{r} and each of the M possible transmitted signals vectors $\{s_m\}$, i.e.,

$$C(r, s_m) = r \cdot s_m = \sum_{k=1}^M r_k s_{mk}, \quad m = 1, 2, \dots, M$$

- ◇ To evaluate the probability of error, let us suppose that the signal s_1 is transmitted. Then the received signal vector is

$$\mathbf{r} = \left[\sqrt{\varepsilon_b} + n_1 \quad n_2 \quad n_3 \cdots n_M \right]$$

where ε_s denotes the symbol energy and n_1, n_2, \dots, n_M are zero-mean, mutually statistically independent Gaussian random variable with equal variance $\sigma_n^2 = \frac{1}{2} N_0$.

Probability of Error for M -ary Orthogonal Signals



- ◇ In this case, the outputs from the bank of M correlations are

$$C(r, s_1) = \sqrt{\varepsilon_s} (\sqrt{\varepsilon_s} + n_1)$$

$$C(r, s_2) = \sqrt{\varepsilon_s} n_2$$

$$\vdots \quad \quad \quad \vdots$$

$$C(r, s_M) = \sqrt{\varepsilon_s} n_M$$

- ◇ Note that the scale factor ε_s may be eliminated from the correlator outputs dividing each output by $\sqrt{\varepsilon_s}$.
- ◇ With this normalization, the PDF of the first correlator output is

$$p_{r_1}(x_1) = \frac{1}{\sqrt{\pi N_0}} \exp \left[-\frac{(x_1 - \sqrt{\varepsilon_s})^2}{N_0} \right]$$

Probability of Error for M -ary Orthogonal Signals



- ◇ And the PDFs of the other $M-1$ correlator outputs are

$$p_{r_m}(x_m) = \frac{1}{\sqrt{\pi N_0}} e^{-x_m^2/N_0}, \quad m = 2, 3, \dots, M$$

- ◇ It is mathematically convenient to first derive the probability that the detector makes a correct decision. This is the probability that r_1 is larger than each of the other $M-1$ correlator outputs n_2, n_3, \dots, n_M . This probability may be expressed as

$$P_c = \int_{-\infty}^{\infty} P(n_2 < r_1, n_3 < r_1, \dots, n_M < r_1 | r_1) p(r_1) dr_1$$

where $P(n_2 < r_1, n_3 < r_1, \dots, n_M < r_1 | r_1)$ denotes the joint probability that n_2, n_3, \dots, n_M are all less than r_1 , conditioned on any given r_1 . Then this joint probability is averaged over all r_1 .

Probability of Error for M-ary Orthogonal Signals



- ◇ Since the $\{r_m\}$ are statistically independent, the joint probability factors into a product of $M-1$ marginal probabilities of the form:

$$\begin{aligned} P(n_m < r_1 | r_1) &= \int_{-\infty}^{r_1} p_{r_m}(x_m) dx_m \quad m = 2, 3, \dots, M \quad \text{--(B)} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1 \sqrt{2/N_0}} e^{-x^2/2} dx \end{aligned} \quad \boxed{x = \sqrt{2/N_0} x_m}$$

- ◇ These probabilities are identical for $m=2, 3, \dots, M$, and, the joint probability under consideration is simply the result in Equation (B) raised to the $(M-1)$ th power. Thus, the probability of a correct decision is

$$P_c = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1 \sqrt{2/N_0}} e^{-x^2/2} dx \right)^{M-1} p(r_1) dr_1$$

Probability of Error for M-ary Orthogonal Signals



- ◇ The probability of a (k -bit) symbol error is

$$P_M = 1 - P_c$$

$$y = r_1 \sqrt{\frac{2}{N_0}}$$

$$P_M = 1 - \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx \right)^{M-1} \exp \left[-\frac{1}{2} \left(y - \sqrt{\frac{2\varepsilon_s}{N_0}} \right)^2 \right] dy \right\}$$

--- (C)

- ◇ The same expression for the probability of error is obtained when any one of the other $M-1$ signals is transmitted. Since all the M signals are equally likely, the expression for P_M given above is the average probability of a symbol error.
- ◇ This expression can be evaluated numerically.

Probability of Error for M -ary Orthogonal Signals



- ◇ In comparing the performance of various digital modulation methods, it is desirable to have the probability of error expressed in terms of the SNR per bit, ε_b/N_0 , instead of the SNR per symbol, ε_s/N_0 .
- ◇ With $M=2^k$, each symbol conveys k bits of information, and hence $\varepsilon_s = k \varepsilon_b$. Thus, Equation (C) may be expressed in terms of ε_b/N_0 by substituting for ε_s .
- ◇ It is also desirable to convert the probability of a symbol error into an equivalent probability of a binary digit error. For equiprobable orthogonal signals, all symbol errors are equiprobable and occur with probability

$$\frac{P_M}{M-1} = \frac{P_b}{2^k-1}$$

Probability of Error for M-ary Orthogonal Signals



- ◆ Furthermore, there are $\binom{k}{n}$ ways in which n bits out of k may be in error. Hence, the average number of bit errors per k -bit symbol is

$$\sum_{n=1}^k n \binom{k}{n} \frac{P_M}{2^k - 1} = k \frac{2^{k-1}}{2^k - 1} P_M \quad \text{--(D)}$$

and the average bit error probability is just the result in Equation (D) divided by k , the number of bits per symbol. Thus,

$$P_b = \frac{2^{k-1}}{2^k - 1} P_M \approx \frac{P_M}{2} \quad k \gg 1$$

- ◆ The graphs of the probability of a binary digit error as a function of the SNR per bit, ε_b/N_0 , are shown in Figure (C) for $M=2,4,8,16,32$, and 64. This figure illustrates that, by increasing the number M of waveforms, one can reduce the SNR per bit required to achieve a given probability of a bit error.

Probability of Error for M-ary Orthogonal Signals



- ◇ For example, to achieve a $P_b=10^{-5}$, the required SNR per bit is a little more than 12dB for $M=2$, but if M is increased to 64 signal waveforms, the required SNR per bit is approximately 6dB. Thus, a savings of over 6dB is realized in transmitter power required to achieve a $P_b=10^{-5}$ by increasing M from $M=2$ to $M=64$.

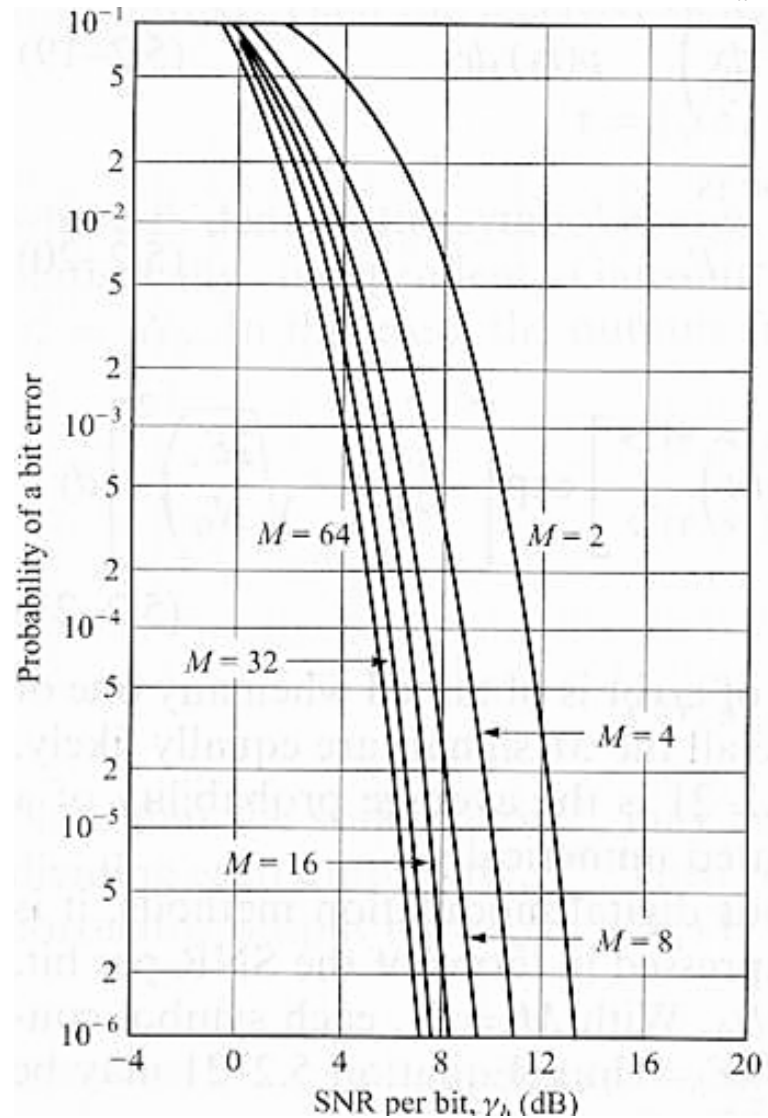


Figure (C)

Probability of Error for M-ary Orthogonal Signals



- ◇ What is the minimum required ε_b/N_0 to achieve an arbitrarily small probability of error as $M \rightarrow \infty$?
- ◇ *A union bound on the probability of error.*
 - ◇ Let us investigate the effect of increasing M on the probability of error for orthogonal signals.
 - ◇ To simplify the mathematical development, we first derive an upper bound on the probability of a symbol error that is much simpler than the exact form given in the following equation:

$$P_M = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx \right)^{M-1} \right] \exp \left[-\frac{1}{2} \left(y - \sqrt{\frac{2\varepsilon_s}{N_0}} \right)^2 \right] dy$$

Probability of Error for M-ary Orthogonal Signals



◇ A union bound on the probability of error (cont.)

- ◇ Recall that the probability of error for binary orthogonal signals is given by:

$$P_b = Q\left(\sqrt{\frac{\varepsilon_b}{N_0}}\right) = Q(\sqrt{\gamma_b})$$

- ◇ Now, if we view the detector for M orthogonal signals as one that makes $M - 1$ binary decisions between the correlator outputs $C(\mathbf{r}, \mathbf{s}_1)$ that contains the signal and the other $M - 1$ correlator outputs $C(\mathbf{r}, \mathbf{s}_m)$, $m=2, 3, \dots, M$, the probability of error is upper-bounded by *union bound* of the $M - 1$ events. That is, if E_i represents the event that $C(\mathbf{r}, \mathbf{s}_i) > C(\mathbf{r}, \mathbf{s}_1)$ for $i \neq 1$, then we have $P_M = P\left(\bigcup_{i=2}^M E_i\right) \leq \sum_{i=2}^M P(E_i)$. Hence,

$$P_M \leq (M - 1) P_b = (M - 1) Q\left(\sqrt{\varepsilon_s / N_0}\right) < M Q\left(\sqrt{\varepsilon_s / N_0}\right)$$

Probability of Error for M-ary Orthogonal Signals



- ◇ A union bound on the probability of error(cont.)
 - ◇ This bound can be simplified further by upper-bounding

$$Q\left(\sqrt{\varepsilon_s / N_0}\right)$$

- ◇ We have

$$Q\left(\sqrt{\varepsilon_s / N_0}\right) < e^{-\varepsilon_s / 2N_0} \quad \text{--(E)}$$

thus,

$$P_M < M e^{-\varepsilon_s / 2N_0} = 2^k e^{-k\varepsilon_b / 2N_0} \quad \text{--(F)}$$

$$P_M < e^{-k(\varepsilon_b / N_0 - 2\ln 2) / 2}$$

- ◇ As $k \rightarrow \infty$ or equivalently, as $M \rightarrow \infty$, the probability of error approaches zero exponentially, provided that ε_b / N_0 is greater than $2\ln 2$,

$$\frac{\varepsilon_b}{N_0} > 2\ln 2 = 1.39 \quad (1.42\text{dB})$$

Probability of Error for M-ary Orthogonal Signals



- ◇ *A union bound on the probability of error*(cont.)
 - ◇ The simple upper bound on the probability of error given by Equation (F) implies that, as long as $\text{SNR} > 1.42$ dB, we can achieve an arbitrarily low P_M .
 - ◇ However, this union bound is not a very tight upper bound as a sufficiently low SNR due to the fact that upper bound for the Q function in Equation (E) is loose.
 - ◇ In fact, by more elaborate bounding techniques, it can be shown that the upper bound in Equation (F) is sufficiently tight for $\varepsilon_b/N_0 > 4 \ln 2$.
 - ◇ For $\varepsilon_b/N_0 < 4 \ln 2$, a tighter upper bound on P_M is

$$P_M < 2e^{-k(\sqrt{\varepsilon_0}/N_0 - \sqrt{\ln 2})^2}$$

Probability of Error for M-ary Orthogonal Signals



- ◇ A union bound on the probability of error(cont.)
 - ◇ Consequently, $P_M \rightarrow 0$ as $k \rightarrow \infty$, provided that

$$\frac{\mathcal{E}_b}{N_0} > \ln 2 = 0.693 \quad (-1.6\text{dB})$$

- ◇ Hence, -1.6 dB is the minimum required SNR per bit to achieve an arbitrarily small probability of error in the limit as $k \rightarrow \infty$ ($M \rightarrow \infty$). This minimum SNR per bit is called the *Shannon limit* for an additive Gaussian noise channel.

Probability of Error for M-ary Biorthogonal Signals



- ◇ As indicated in Chapter 3, a set of $M=2^k$ biorthogonal signals are constructed from $\frac{1}{2} M$ orthogonal signals by including the negatives of the orthogonal signals. Thus, we achieve a reduction in the complexity of the demodulator for the biorthogonal signals relative to that for orthogonal signals, since the former is implemented with $\frac{1}{2} M$ cross correlation or matched filters, whereas the latter required M matched filters or cross correlators.
- ◇ Let us assume that the signal $s_1(t)$ corresponding to the vector $s_1 = [\sqrt{\epsilon_s} \ 0 \ 0 \cdots 0]$ was transmitted. The received signal vector is

$$\mathbf{r} = [\sqrt{\epsilon_s} + n_1 \ n_2 \ \cdots \ n_{M/2}]$$

where the $\{n_m\}$ are zero-mean, mutually statistically independent and identically distributed Gaussian random variables with variance $\sigma_n^2 = \frac{1}{2} N_0$.

Probability of Error for M-ary Biorthogonal Signals



- ◇ The optimum detector decides in favor of the signal corresponding to the largest in magnitude of the cross correlators

$$C(\mathbf{r}, \mathbf{s}_m) = \mathbf{r} \cdot \mathbf{s}_m = \sum_{k=1}^{M/2} r_k s_{mk}, \quad m = 1, 2, \dots, \frac{1}{2}M$$

while the sign of this largest term is used to decide whether $s_m(t)$ or $-s_m(t)$ was transmitted.

- ◇ According to this decision rule, the probability of a correct decision is equal to the probability that $r_1 = \sqrt{\varepsilon_s} + n_1 > 0$ and r_1 exceeds $|r_m| = |n_m|$ for $m=2, 3, \dots, \frac{1}{2}M$. But

$$P(|n_m| < r_1 | r_1 > 0) = \frac{1}{\sqrt{\pi N_0}} \int_{-r_1}^{r_1} e^{-x^2/N_0} dx = \frac{1}{\sqrt{2\pi}} \int_{-r_1/\sqrt{N_0/2}}^{r_1/\sqrt{N_0/2}} e^{-y^2/2} dy$$

$$y = \sqrt{\frac{N_0}{2}} x$$

Probability of Error for M-ary Biorthogonal Signals



- ◇ Then, the probability of a correct decision is

$$P_c = \int_0^\infty \left(\frac{1}{\sqrt{2\pi}} \int_{-r_1/\sqrt{N_0/2}}^{r_1/\sqrt{N_0/2}} e^{-x^2/2} \right)^{M/2-1} p(r_1) dr_1$$

from which, upon substitution for $p(r_1)$, we obtain

$$P_c = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2\varepsilon_s/N_0}}^\infty \left(\frac{1}{\sqrt{2\pi}} \int_{-(v+\sqrt{2\varepsilon_s/N_0})}^{v+\sqrt{2\varepsilon_s/N_0}} e^{-x^2/2} dx \right)^{M/2-1} e^{-v^2/2} dv \quad \text{--(G)}$$

$$p_{r_1}(x_1) = \frac{1}{\sqrt{\pi N_0}} \exp \left[-\frac{(x_1 - \sqrt{\varepsilon_s})^2}{N_0} \right] \quad v = (r_1 - \sqrt{\varepsilon_s}) / \sqrt{\frac{N_0}{2}}$$

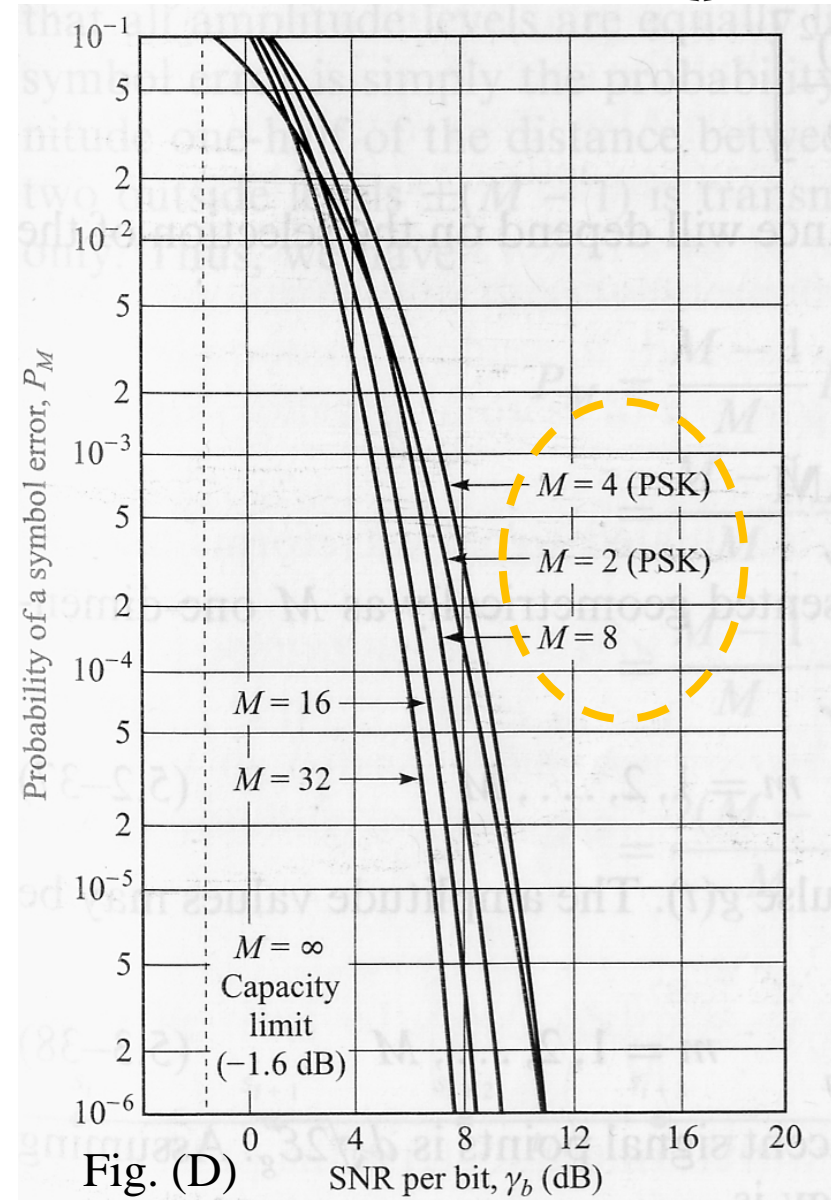
where we have used the PDF of r_1 given in Page 63.

- ◇ The probability of a symbol error $P_M = 1 - P_c$

Probability of Error for M-ary Biorthogonal Signals



- ◇ P_c , and P_M may be evaluated numerically for different values of M . The graph shown in the following figure (D) illustrates P_M as a function of ε_b/N_0 , where $\varepsilon_s = k \varepsilon_b$, for $M=2,4,8,16$ and 32.



Probability of Error for M -ary Biorthogonal Signals



- ◇ In this case, the probability of error for $M=4$ is greater than that for $M=2$. This is due to the fact that we have plotted the symbol error probability P_M in Figure(D).
- ◇ If we plotted the equivalent bit error probability, we should find that the graphs for $M=2$ and $M=4$ coincide. (Why?)
- ◇ As in the case of orthogonal signals, as $M \rightarrow \infty$ ($k \rightarrow \infty$), the minimum required ε_b/N_0 to achieve an arbitrarily small probability of error is -1.6 dB, the Shannon limit.

Probability of Error for M -ary PAM



- Recall that M -ary PAM signals are represented geometrically as M one-dimensional signal points with value :

$$s_m = \sqrt{\frac{1}{2} \mathcal{E}_g} A_m \quad m = 1, 2, \dots, M$$

where $A_m = (2m - 1 - M)d$, $m = 1, 2, \dots, M$

- The Euclidean distance between adjacent signal points is $d\sqrt{2\mathcal{E}_g}$.
- Assuming equally probable signals, the average energy is :

$$\mathcal{E}_{av} = \frac{1}{M} \sum_{m=1}^M \mathcal{E}_m = \frac{1}{M} \sum_{m=1}^M s_m^2 = \frac{d^2 \mathcal{E}_g}{2M} \sum_{m=1}^M (2m - 1 - M)^2$$

$$= \frac{d^2 \mathcal{E}_g}{2M} \left[\frac{1}{3} M (M^2 - 1) \right] = \frac{1}{6} (M^2 - 1) d^2 \mathcal{E}_g$$

$\sum_{m=1}^M m = \frac{M(M+1)}{2}$ $\sum_{m=1}^M m^2 = \frac{M(M+1)(2M+1)}{6}$

Probability of Error for M -ary PAM

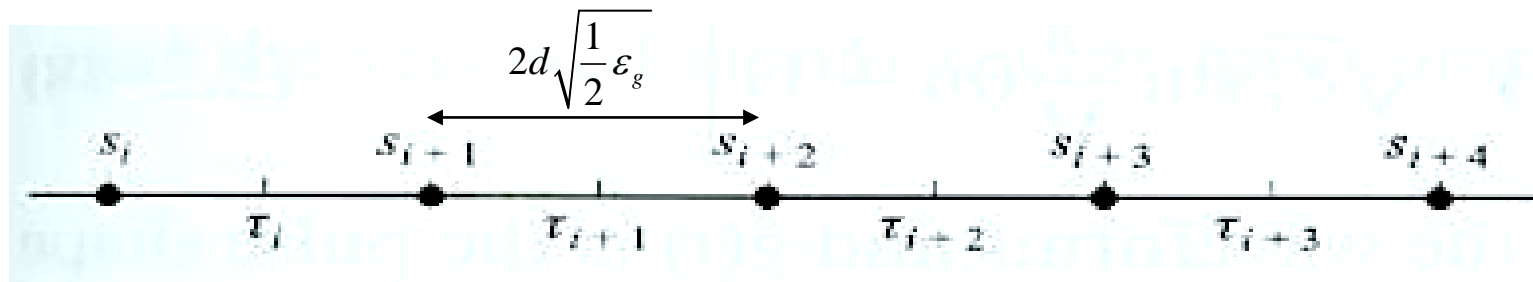


- ◇ Equivalently, we may characterize these signals in terms of their average power, where is :

$$P_{av} = \frac{\mathcal{E}_{av}}{T} = \frac{1}{6} (M^2 - 1) \frac{d^2 \mathcal{E}_g}{T} \quad (*)$$

- ◇ The average probability of error for M -ary PAM :

- ◇ The detector compares the demodulator output r with a set of $M-1$ thresholds, which are placed at the midpoints of successive amplitude level and decision is made in favor of the amplitude level that is close to r .



- ◇ We note that if the m th amplitude level is transmitted, the demodulator output is

$$r = s_m + n = \sqrt{\frac{1}{2} \mathcal{E}_g} A_m + n$$

where the noise variable n has zero-mean and variance $\sigma_n^2 = \frac{1}{2} N_0$

Probability of Error for M -ary PAM



- Assuming all amplitude levels are equally likely a priori, the average probability of a symbol error is the probability that the noise variable n exceeds in magnitude one-half of the distance between levels.
- However, when either one of the two outside levels $\pm(M-1)$ is transmitted, an error can occur in one direction only.
- As a result, we have the error probability:

$$\begin{aligned}
 P_M &= \frac{M-1}{M} P\left(|r - s_m| > d \sqrt{\frac{1}{2} \mathcal{E}_g}\right) = \frac{M-1}{M} \frac{2}{\sqrt{2\pi N_0/2}} \int_{d\sqrt{\mathcal{E}_g/2}}^{\infty} e^{-x^2/N_0} dx \\
 &= \frac{M-1}{M} \frac{2}{\sqrt{2\pi}} \int_{\sqrt{d^2 \mathcal{E}_g / N_0}}^{\infty} e^{-y^2/2} dy \quad \text{where } y = x / \sqrt{N_0/2} \\
 &= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{d^2 \mathcal{E}_g}{N_0}}\right)
 \end{aligned}$$

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2} dx \quad t \geq 0$$

$$\frac{1}{M} \cdot (M-2) + \frac{1}{M} \cdot \frac{1}{2} \cdot 2$$

Probability of Error for M -ary PAM



- ◇ From (*) in page 81, we note that $d^2 \mathcal{E}_g = \frac{6}{M^2 - 1} P_{av} T$
- ◇ By substituting for $d^2 \mathcal{E}_g$, we obtain the average probability of a symbol error for PAM in terms of the average power :

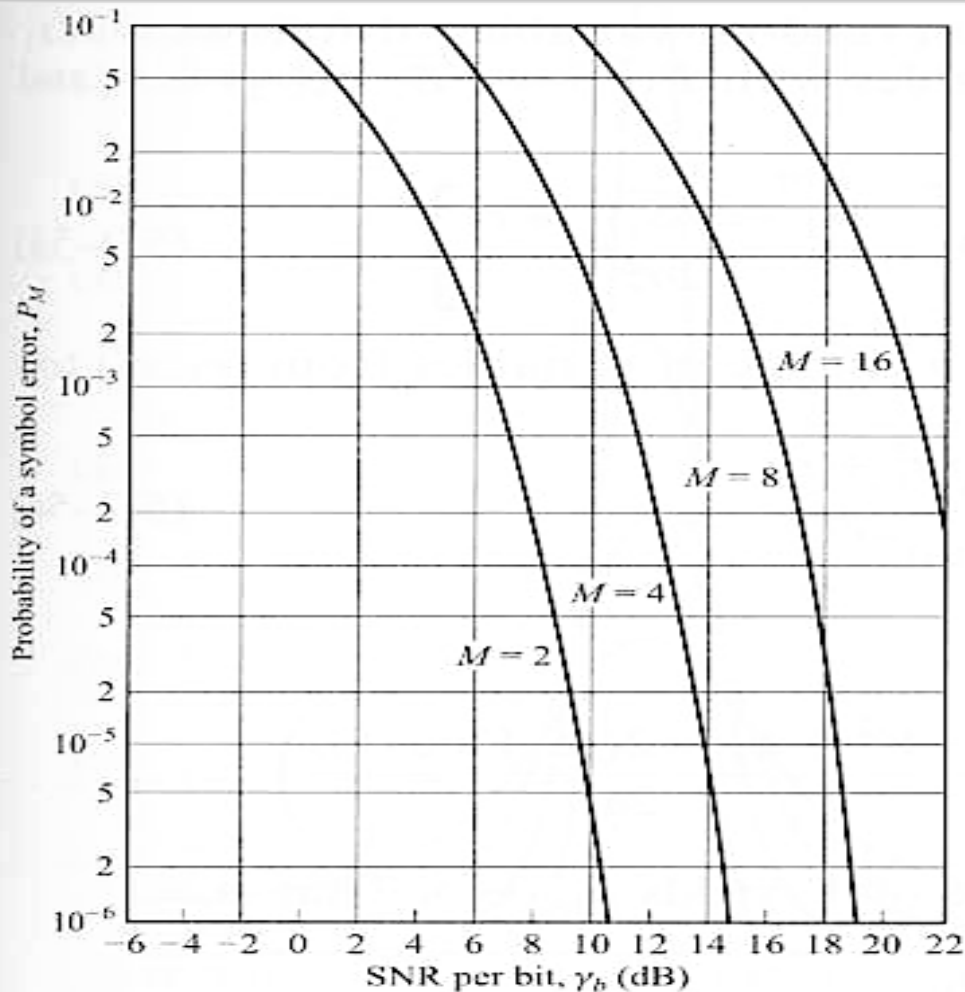
$$P_M = \frac{2(M-1)}{M} Q \left(\sqrt{\frac{6P_{av}T}{(M^2-1)N_0}} \right) = \frac{2(M-1)}{M} Q \left(\sqrt{\frac{6\mathcal{E}_{av}}{(M^2-1)N_0}} \right)$$

- ◇ It is customary for us to use the SNR per bit as the basic parameter, and since $T = kT_b$ and $k = \log_2 M$:

$$P_M = \frac{2(M-1)}{M} Q \left(\sqrt{\frac{(6 \log_2 M) \mathcal{E}_{bav}}{(M^2-1)N_0}} \right)$$

where $\mathcal{E}_{bav} = P_{av} T_b$ is the average bit energy and \mathcal{E}_{bav}/N_0 is the average SNR per bit.

Probability of Error for M -ary PAM



Probability of a symbol error for PAM

- ◇ The case $M=2$ corresponds to the error probability for binary antipodal signals.
- ◇ The SNR per bit increase by over 4 dB for every factor-of-2 increase in M .
- ◇ For large M , the additional SNR per bit required to increase M by a factor of 2 approaches 6 dB.

Probability of Error for M -ary PSK



- ◇ Recall from that digital phase-modulated signal waveforms may be expressed as:

$$s_m(t) = g(t) \cos \left[2\pi f_c t + \frac{2\pi}{M} (m-1) \right], \quad 1 \leq m \leq M, \quad 0 \leq t \leq T$$

and have the vector representation:

Energy in each waveform.

$$s_m(t) = \left[\sqrt{\mathcal{E}_s} \cos \frac{2\pi}{M} (m-1) \quad \sqrt{\mathcal{E}_s} \sin \frac{2\pi}{M} (m-1) \right], \quad \mathcal{E}_s = \frac{1}{2} \mathcal{E}_g$$

- ◇ Since the signal waveforms have equal energy, the optimum detector for the AWGN channel computes the correlation metrics

$$C(\mathbf{r}, \mathbf{s}_m) = \mathbf{r} \cdot \mathbf{s}_m, \quad m = 1, 2, \dots, M$$

Probability of Error for M -ary PSK



- ◇ In other word, the received signal vector $\mathbf{r}=[r_1 \ r_2]$ is projected onto each of the M possible signal vectors and a decision is made in favor of the signal with the largest projection.
- ◇ This correlation detector is equivalent to a phase detector that computes the phase of the received signal from \mathbf{r} .
- ◇ We selects the signal vector \mathbf{s}_m whose phase is closer to \mathbf{r} .
- ◇ The phase of \mathbf{r} is

$$\Theta_r = \tan^{-1} \frac{r_2}{r_1}$$

- ◇ We will determine the PDF of Θ_r , and compute the probability of error from it.
- ◇ Consider the case in which the transmitted signal phase is $\Theta_r = 0$, corresponding to the signal $s_1(t)$.

Probability of Error for M -ary PSK



- ◇ The transmitted signal vector is $s_1 = [\sqrt{\mathcal{E}_s} \ 0]$, and the received signal vector has components:

$$r_1 = \sqrt{\mathcal{E}_s} + n_1 \quad r_2 = n_2$$

- ◇ Because n_1 and n_2 are jointly Gaussian random variable, it follow that r_1 and r_2 are jointly Gaussian random variable variables with

$$E(r_1) = \sqrt{\mathcal{E}_s}, E(r_2) = 0, \text{ and } \sigma_{r_1}^2 = \sigma_{r_2}^2 = \frac{1}{2} N_0 = \sigma_r^2.$$

$$p_r(r_1, r_2) = \frac{1}{2\pi\sigma_r^2} \exp\left[-\frac{(r_1 - \sqrt{\mathcal{E}_s})^2 + r_2^2}{2\sigma_r^2}\right]$$

- ◇ The PDF of the phase Θ_r is obtained by a change in variables from (r_1, r_2) to :

$$V = \sqrt{r_1^2 + r_2^2} \quad \Theta_r = \tan^{-1} \frac{r_2}{r_1}$$

Probability of Error for M -ary PSK



- ◆ The joint PDF of V and Θ_r :

$$dr_1 dr_2 = V dV d\Theta$$

$$p_{V, \Theta_r}(V, \Theta_r) = \frac{V}{2\pi\sigma_r^2} \exp\left(-\frac{V^2 + \mathcal{E}_s - 2\sqrt{\mathcal{E}_s} V \cos \Theta_r}{2\sigma_r^2}\right)$$

- ◆ Integration of $p_{V, \Theta_r}(V, \Theta_r)$ over the range of V yields $p_{\Theta_r}(\Theta_r)$

$$p_{\Theta_r}(\Theta_r) = \int_0^{\infty} p_{V, \Theta_r}(V, \Theta_r) dV$$

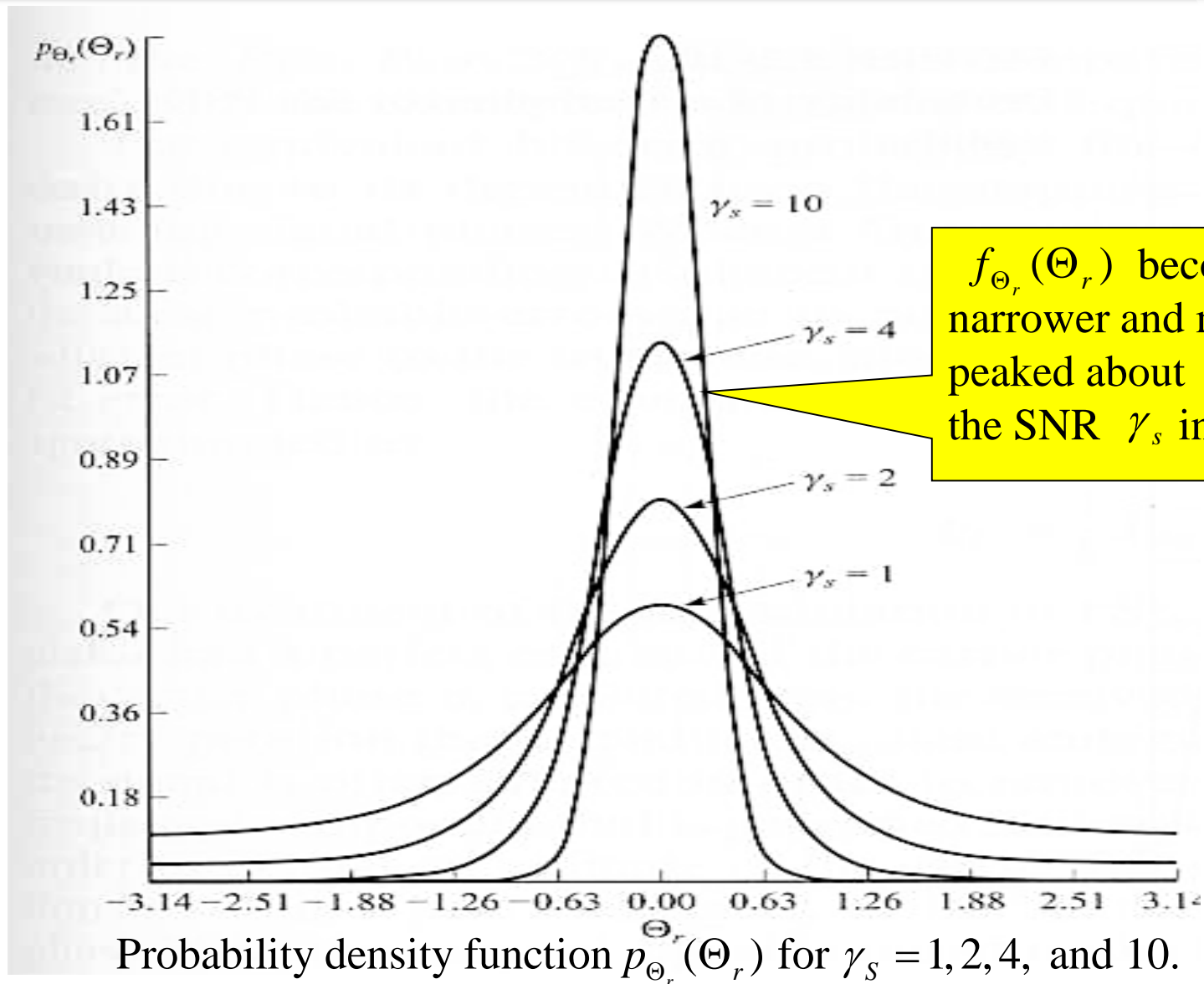
$$= \frac{1}{2\pi\sigma_r^2} e^{-\mathcal{E}_s/N_0 \sin^2 \Theta_r} \int_0^{\infty} \frac{V}{N_0/2} e^{-\left(\frac{V}{N_0/2} - \sqrt{2\mathcal{E}_s/N_0} \cos \Theta_r\right)^2 / 2\sigma_r^2} dV$$

$$= \frac{1}{2\pi\sigma_r^2} e^{-\gamma_s \sin^2 \Theta_r} \int_0^{\infty} V' e^{-(V' - \sqrt{2\gamma_s} \cos \Theta_r)^2 / 2\sigma_r^2} dV'$$

$$V' = \frac{V}{N_0/2}$$

where we define the symbol SNR as $\gamma_s = \mathcal{E}_s/N_0$.

Probability of Error for M -ary PSK



$f_{\Theta_r}(\Theta_r)$ becomes narrower and more peaked about $\Theta_r = 0$ as the SNR γ_s increases.

Probability of Error for M -ary PSK



- ◇ When $s_1(t)$ is transmitted, a decision error is made if the noise causes the phase to fall outside the range $-\pi/M \leq \Theta_r \leq \pi/M$. Hence, the probability of a symbol error is

$$P_M = 1 - \int_{-\pi/M}^{\pi/M} p_{\Theta_r}(\Theta_r) d\Theta_r$$

- ◇ In general, the integral of $p_{\Theta_r}(\Theta_r)$ doesn't reduce to a simple form and must be evaluated numerically, except for $M=2$ and $M=4$.
- ◇ For binary phase modulation, the two signals $s_1(t)$ and $s_2(t)$ are antipodal. Hence, the error probability is

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

Probability of Error for M -ary PSK



- ◇ When $M = 4$, we have in effect two binary phase-modulation signals in phase quadrature.
- ◇ Since there is no crosstalk or interference between the signals on the two quadrature carriers, the bit error probability is identical to that of $M = 2$.

- ◇ Then the symbol error probability for $M=4$ is determined by noting that

$$P_c = (1 - P_2)^2 = \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \right]^2$$

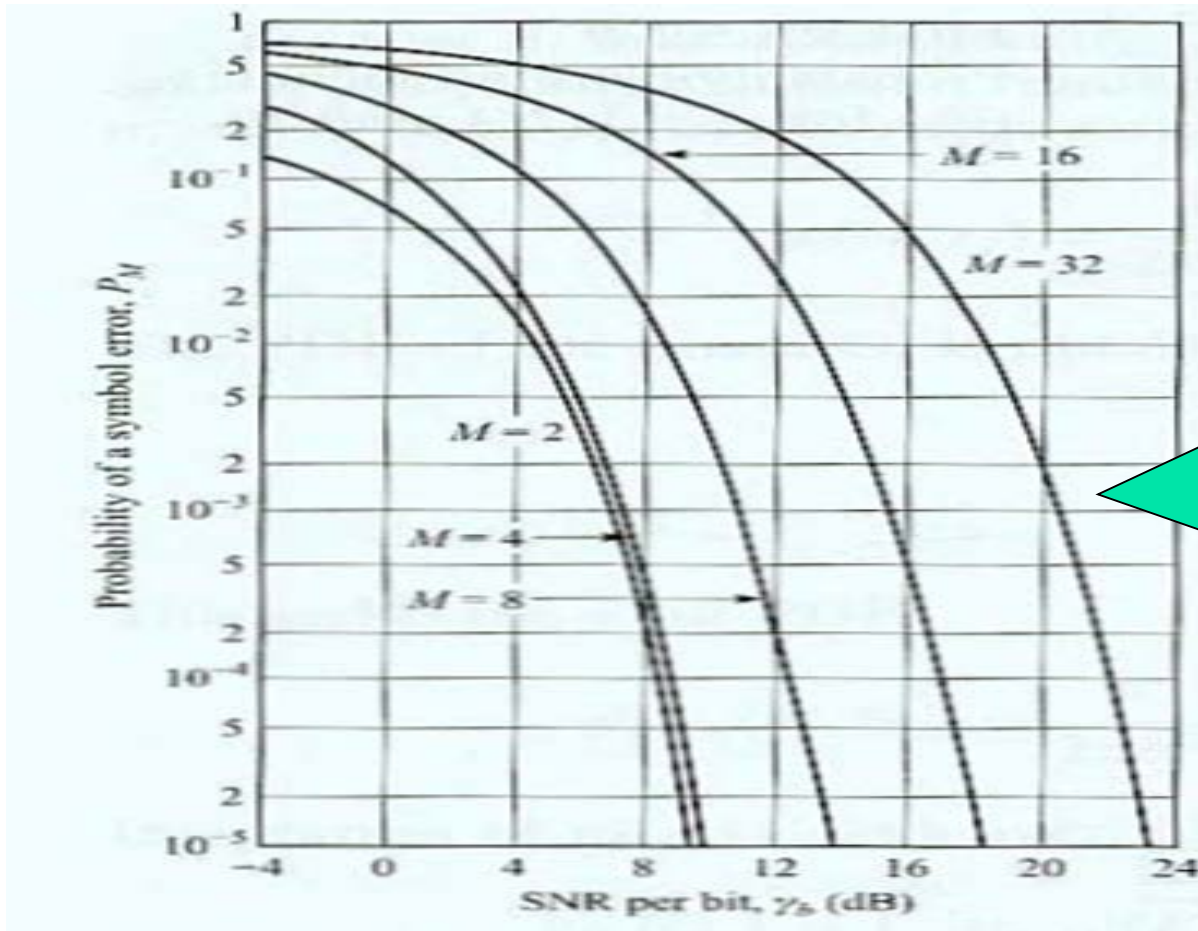
where P_c is the probability of a correct decision for the 2-bit symbol.

- ◇ There, the symbol error probability for $M = 4$ is

$$P_4 = 1 - P_c = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \right]$$

Probability of Error for M -ary PSK

- For $M > 4$, the symbol error probability P_M is obtained by numerically integrating Equation
$$P_M = 1 - \int_{-\pi/M}^{\pi/M} p_{\Theta_r}(\Theta_r) d\Theta_r$$



For large values of M , doubling the number of phases requires an additional 6 dB/bit to achieve the same performance

Probability of Error for M -ary PSK



◇ An approximation to the error probability for large M and for large SNR may be obtained by first approximating $p_{\Theta_r}(\Theta)$.

◇ For $\mathcal{E}_s/N_0 \gg 1$ and $|\Theta_r| \leq 0.5\pi$, $p_{\Theta_r}(\Theta)$ is well approximated as :

$$P_{\Theta_r}(\Theta_r) \approx \sqrt{\frac{\gamma_s}{\pi}} \cos \Theta_r e^{-\gamma_s \sin^2 \Theta_r}$$

◇ Performing the change in variable from Θ_r to $u = \sqrt{\mathcal{V}_s} \sin \Theta_r$,

$$\begin{aligned} P_M &\approx 1 - \int_{-\pi/M}^{\pi/M} \sqrt{\frac{\gamma_s}{\pi}} \cos \Theta_r e^{-\gamma_s \sin^2 \Theta_r} d\Theta_r \\ &\approx \frac{2}{\sqrt{\pi}} \int_{\sqrt{2\gamma_s} \sin(\pi/M)}^{\infty} e^{-u^2} du \\ &= 2Q\left(\sqrt{2\gamma_s} \sin \frac{\pi}{M}\right) = 2Q\left(\sqrt{2k\gamma_b} \sin \frac{\pi}{M}\right) \end{aligned}$$

$$k = \log_2 M$$

$$\mathcal{V}_s = k \mathcal{V}_b$$

Probability of Error for M -ary PSK



- ◇ When a Gray Code is used in the mapping of k -bits symbols into the corresponding signal phases, two k -bit symbols corresponding to adjacent signal phases differ in only a signal bit.
- ◇ The most probable error result in the erroneous selection of an adjacent phase to the true one.
- ◇ Most k -bit symbol error contain only a single-bit error. The equivalent bit error probability for M -ary PSK is well approximated as :

$$P_b \approx \frac{1}{k} P_M$$

- ◇ In practice, the carrier phase is extracted from the received signal by performing some nonlinear operation that introduces a *phase ambiguity*.

Probability of Error for M -ary PSK



- ◇ For binary PSK, the signal is often squared in order to remove the modulation, and the double-frequency component that is generated is filtered and divided by 2 in frequency in order to extract an estimate of the carrier frequency and phase φ .
 - ◇ This operation result in a phase ambiguity of 180° in the carrier phase.
- ◇ For four-phase PSK, the received signal is raised to the fourth power to remove the digital modulation, and the resulting fourth harmonic of the carrier frequency is filtered and divided by 4 in order to extract the carrier component.
 - ◇ These operations yield a carrier frequency component containing φ , but there are phase ambiguities of $\pm 90^\circ$ and 180° in the phase estimate.
- ◇ Consequently, we do not have an absolute estimate of the carrier phase for demodulation.

Probability of Error for M -ary PSK



- ◇ The phase ambiguity problem can be overcome by encoding the information in *phase differences* between successive signal transmissions as opposed to absolute phase encoding.
 - ◇ For example, in binary PSK, the information bit 1 may be transmitted by shifting the phase of carrier by 180° relative to the previous carrier phase. Bit 0 is transmitted by a zero phase shift relative to the phase in the previous signaling interval.
 - ◇ In four-phase PSK, the relative phase shifts between successive intervals are 0 , 90° , 180° , and -90° , corresponding to the information bits 00, 01, 11, and 10, respectively.
- ◇ The PSK signals resulting from the encoding process are said to be *differentially encoded*.

Probability of Error for M -ary PSK



- ◇ The detector is a relatively simple phase comparator that compares the phase of the demodulated signal over two consecutive intervals to extract the information.
- ◇ Coherent demodulation of differentially encoded PSK results in a higher probability of error than that derived for absolute phase encoding.
- ◇ With differentially encoded PSK, an error in the demodulated phase of the signal in any given interval will usually result in decoding errors over two consecutive signaling intervals.
- ◇ The probability of error in differentially encoded M -ary PSK is approximately twice the probability of error for M -ary PSK with absolute phase encoding.

Differential PSK (DPSK) and Its Performance



- ◇ The received signal of a differentially encoded phase-modulated signal in any given signaling interval is compared to the phase of the received signal from the preceding signaling interval.
- ◇ We demodulate the differentially encoded signal by multiplying $r(t)$ by $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$ integrating the two products over the interval T .
- ◇ At the k th signaling interval, the demodulator output :

$$\mathbf{r}_k = \left[\sqrt{\mathcal{E}_s} \cos(\theta_k - \phi) + n_{k1} \quad \sqrt{\mathcal{E}_s} \sin(\theta_k - \phi) + n_{k2} \right]$$

or equivalently,

$$r_k = \sqrt{\mathcal{E}_s} e^{j(\theta_k - \phi)} + n_k$$

where θ_k is the phase angle of the transmitted signal at the k th signaling interval, ϕ is the carrier phase, and $n_k = n_{k1} + jn_{k2}$ is the noise vector.

Differential PSK (DPSK) and Its Performance



- ◇ Similarly, the received signal vector at the output of the demodulator in the preceding signaling interval is :

$$r_{k-1} = \sqrt{\mathcal{E}_s} e^{j(\theta_{k-1}-\phi)} + n_{k-1}$$

- ◇ The decision variable for the phase detector is the phase difference between these two complex numbers. Equivalently, we can project r_k onto r_{k-1} and use the phase of the resulting complex number :

$$r_k r_{k-1}^* = \mathcal{E}_s e^{j(\theta_k - \theta_{k-1})} + \sqrt{\mathcal{E}_s} e^{j(\theta_k - \phi)} n_{k-1}^* + \sqrt{\mathcal{E}_s} e^{-j(\theta_{k-1} - \phi)} n_k + n_k n_{k-1}^*$$

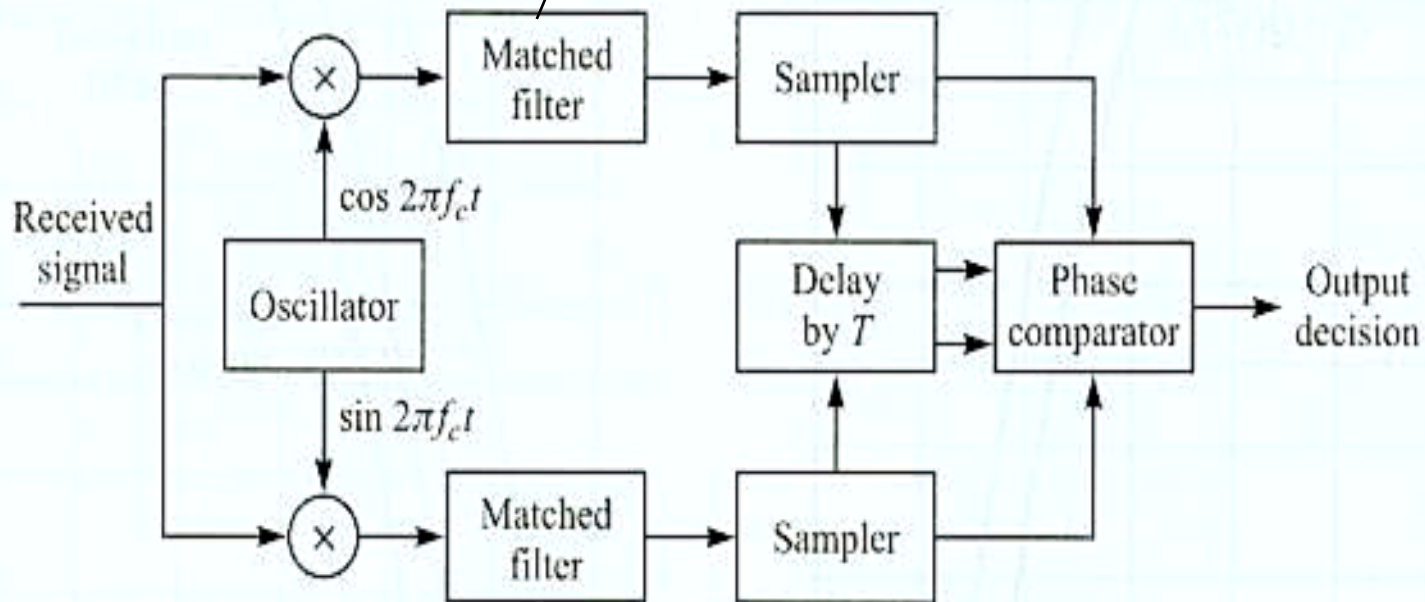
which, in the absence of noise, yields the phase difference $\theta_k - \theta_{k-1}$.

- ◇ Differentially encoded PSK signaling that is demodulated and detected as described above is called *differential PSK* (DPSK).

Differential PSK (DPSK) and Its Performance



If the pulse $g(t)$ is rectangular, the matched filter may be replaced by integrate-and-dump filter



Block diagram of DPSK demodulator

Differential PSK (DPSK) and Its Performance



- ◇ The error probability performance of a DPSK demodulator and detector
 - ◇ The derivation of the exact value of the probability of error for M -ary DPSK is extremely difficult, except for $M = 2$.
 - ◇ Without loss of generality, suppose the phase difference $\theta_k - \theta_{k-1} = 0$. Furthermore, the exponential factor $e^{-j(\theta_{k-1} - \phi)}$ and $e^{j(\theta_k - \phi)}$ can be absorbed into Gaussian noise components n_{k-1} and n_k , without changing their statistical properties.
$$r_k r_{k-1}^* = \mathcal{E}_s + \sqrt{\mathcal{E}_s} (n_k + n_{k-1}^*) + n_k n_{k-1}^*$$
- ◇ The complication in determining the PDF of the phase is the term $n_k n_{k-1}^*$.

Differential PSK (DPSK) and Its Performance



- ◇ However, at SNRs of practical interest, the term $n_k n_{k-1}^*$ is small relative to the dominant noise term $\sqrt{\mathcal{E}_s}(n_k + n_{k-1}^*)$.
- ◇ We neglect the term $n_k n_{k-1}^*$ and normalize $r_k r_{k-1}^*$ by dividing through by $\sqrt{\mathcal{E}_s}$, the new set of decision metrics becomes :

$$\begin{aligned}x &= \sqrt{\mathcal{E}_s} + \text{Re}(n_k + n_{k-1}^*) \\y &= \text{Im}(n_k + n_{k-1}^*)\end{aligned}$$

- ◇ The variables x and y are uncorrelated Gaussian random variable with identical variances $\sigma_n^2 = N_0$. The phase is

$$\Theta_r = \tan^{-1} \frac{y}{x}$$

- ◇ The noise variance is now twice as large as in the case of PSK. Thus we can conclude that the performance of DPSK is 3 dB poorer than that for PSK.

Differential PSK (DPSK) and Its Performance



- ◇ This result is relatively good for $M \geq 4$, but it is pessimistic for $M = 2$ that the loss in binary DPSK relative to binary PSK is less than 3 dB at large SNR.
 - ◇ In binary DPSK, the two possible transmitted phase differences are 0 and π rad. Consequently, only the real part of $r_k r_{k-1}^*$ is needed for recovering the information.

$$\text{Re}(r_k r_{k-1}^*) = \frac{1}{2} (r_k r_{k-1}^* + r_k^* r_{k-1})$$

- ◇ Because the phase difference between the two successive signaling intervals is zero, an error is made if $\text{Re}(r_k r_{k-1}^*) < 0$.
- ◇ The probability that $r_k r_{k-1}^* + r_k^* r_{k-1} < 0$ is a special case of a derivation, given in Appendix B (Proakis 4th Edition).

Differential PSK (DPSK) and Its Performance



- ◇ Appendix B concerned with the probability that a general quadratic form in complex-valued Gaussian random Variable is less than zero. According to Equation B-21, we find it depend entirely on the first and second moments of the complex-valued Gaussian random variables r_k and r_{k-1} .
- ◇ We obtain the probability of error for binary DPSK in the form

$$P_b = \frac{1}{2} e^{-\mathcal{E}_b/N_0}$$

where \mathcal{E}_b/N_0 is the SNR per bit.

Differential PSK (DPSK) and Its Performance



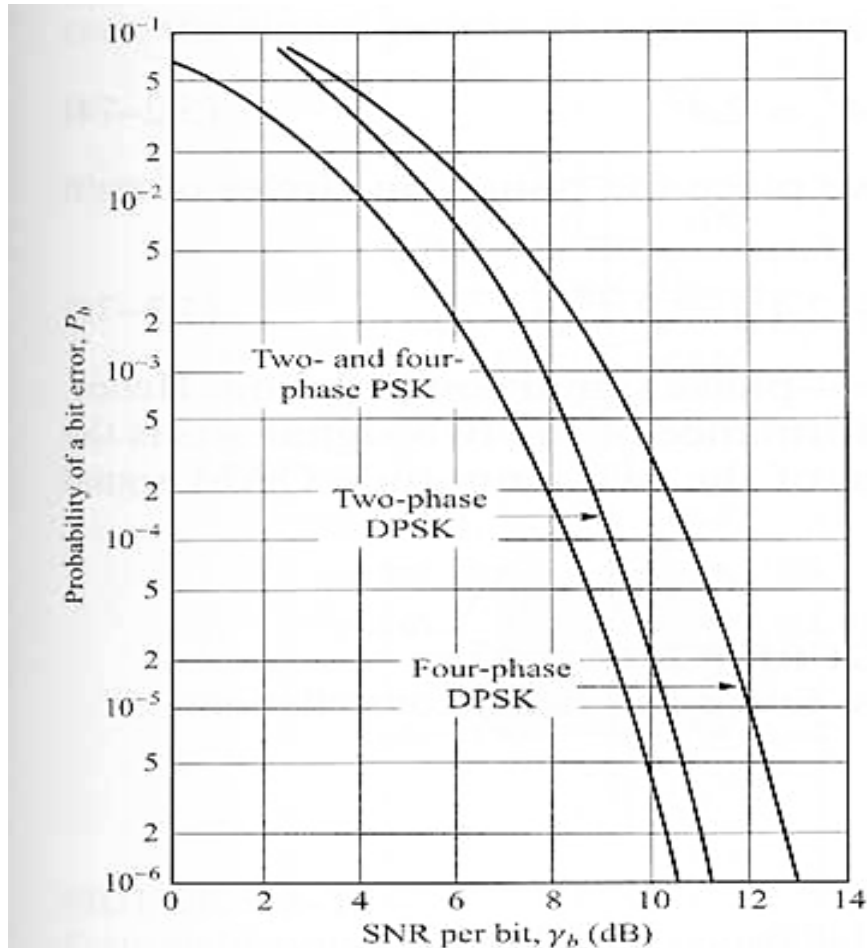
- ◇ The probability of a binary digit error for four-phase DPSK with Gray coding can be expressed in terms of well-known functions, but its derivation is quite involved.
 - ◇ According to Appendix C, it is expressed in the form :

$$P_b = Q_1(a, b) - \frac{1}{2} I_0(ab) \exp\left[-\frac{1}{2}(a^2 + b^2)\right]$$

where $Q_1(a, b)$ is the Marcum Q function, $I_0(x)$ is the modified Bessel function of order zero, and the parameters a and b are defined as

$$a = \sqrt{2\gamma_b \left(1 - \sqrt{\frac{1}{2}}\right)}, \text{ and } b = \sqrt{2\gamma_b \left(1 + \sqrt{\frac{1}{2}}\right)}$$

Differential PSK (DPSK) and Its Performance



- ◇ Because binary DPSK is only slightly inferior to binary PSK at large SNR, and DPSK does not require an elaborate method for estimate the carrier phase, it is often used in digital communication system.

Probability of bit error for binary and four-phase PSK and DPSK

Probability of Error for QAM



- ◇ Recall that QAM signal waveforms may be expressed as

$$s_m(t) = A_{mc} g(t) \cos 2\pi f_c t - A_{ms}(t) \sin 2\pi f_c t$$

where A_{mc} and A_{ms} are the information-bearing signal amplitudes of the quadrature carriers and $g(t)$ is the signal pulse.

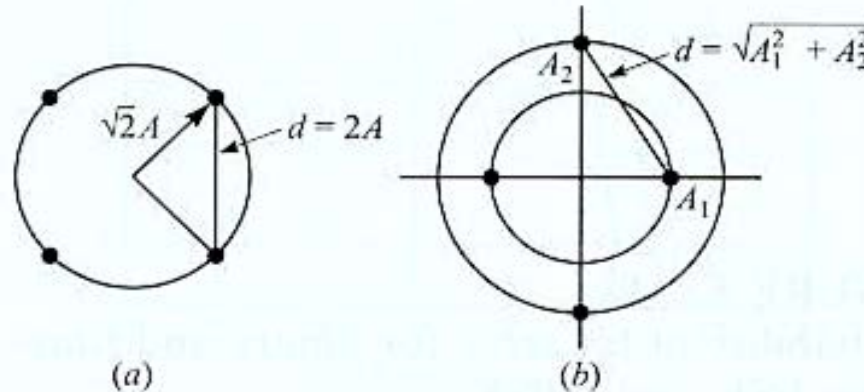
- ◇ The vector representation of these waveform is

$$\mathbf{s}_m = \left[A_{mc} \sqrt{\frac{1}{2}} \mathcal{E}_g \quad A_{ms} \sqrt{\frac{1}{2}} \mathcal{E}_g \right]$$

- ◇ To determine the probability of error for QAM, we must specify the signal point constellation.

Probability of Error for QAM

- ◇ QAM signal sets that have $M = 4$ points.



- ◇ Figure (a) is a four-phase modulated signal and Figure (b) is with two amplitude levels, labeled A_1 and A_2 , and four phases.
- ◇ Because the probability of error is dominated by the minimum distance between pairs of signal points, let us impose the condition that $d_{\min}^{(e)} = 2A$ and we evaluate the average transmitter power, based on the premise that all signal points are equally probable.

Probability of Error for QAM



- ◇ For the four-phase signal, we have

$$P_{av} = \frac{1}{4}(4)2A^2 = 2A^2$$

- ◇ For the two-amplitude, four-phase QAM, we place the points on circles of radii A and $\sqrt{3}A$. Thus, $d_{\min}^{(e)} = 2A$, and

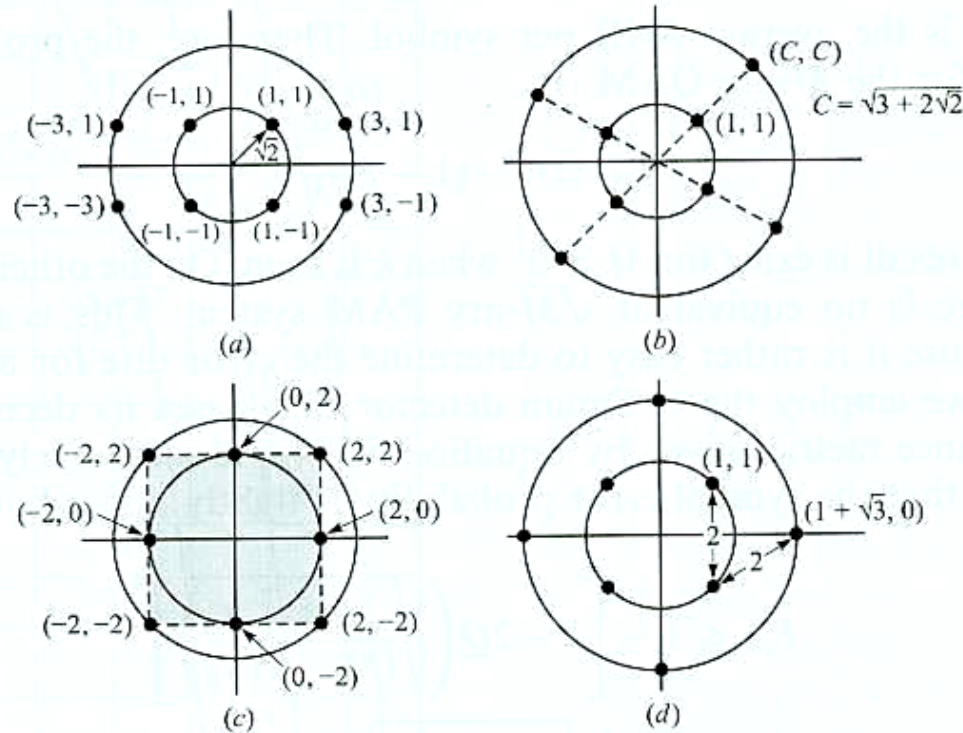
$$P_{av} = \frac{1}{2} [2(3A^2) + 2A^2] = 2A^2$$

which is the same average power as the $M = 4$ -phase signal constellation.

- ◇ Hence, for all practical purposes, the error rate performance of the two signal sets is the same.
- ◇ There is no advantage of the two-amplitude QAM signal set over $M = 4$ -phase modulation.

Probability of Error for QAM

- ◇ QAM signal sets that have $M = 8$ points.
 - ◇ We consider the four signal constellations :



Assuming that the signal points are equally probable, the average transmitted signal power is :

$$\begin{aligned}
 P_{av} &= \frac{1}{M} \sum_{m=1}^M (A_{mc}^2 + A_{ms}^2) \\
 &= \frac{A^2}{M} \sum_{m=1}^M (a_{mc}^2 + a_{ms}^2)
 \end{aligned}$$

The coordinates (A_{mc}, A_{ms}) for each signal point are normalized by A

Probability of Error for QAM

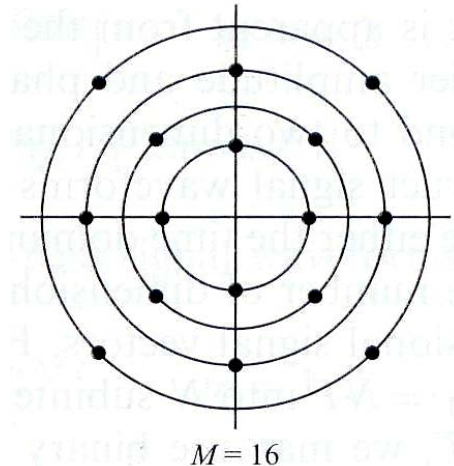


- ◇ The two sets (a) and (c) contain signal points that fall on a rectangular grid and have $P_{av} = 6A^2$.
- ◇ The signal set (b) requires an average transmitted power $P_{av} = 6.83A^2$, and (d) requires $P_{av} = 4.73A^2$.
- ◇ The fourth signal set (d) requires approximately 1 dB less power than the first two and 1.6 dB less power than the third to achieve the same probability of error.
- ◇ The fourth signal constellation is known to be the best eight-point QAM constellation because it requires the least power for a given minimum distance between signal points.

Probability of Error for QAM



- ◇ QAM signal sets for $M \geq 16$
 - ◇ For 16-QAM, the signal points at a given amplitude level are phase-rotated by relative to the signal points at adjacent amplitude levels.
 - ◇ However, the circular 16-QAM constellation is not the best 16-point QAM signal constellation for the AWGN channel.



- ◇ Rectangular M -ary QAM signal are most frequently used in practice. The reasons are :
 - ◇ Rectangular QAM signal constellations have the distinct advantage of being easily generated as two PAM signals impressed on phase-quadrature carriers.

Probability of Error for QAM



- ◇ The average transmitted power required to achieve a given minimum distance is only slightly greater than that of the best M -ary QAM signal constellation.
- ◇ For rectangular signal constellations in which $M = 2^k$, where k is even, the QAM signal constellation is equivalent to two PAM signals on quadrature carriers, each having $\sqrt{M} = 2^{k/2}$ signal points.
- ◇ The probability of error for QAM is easily determined from the probability of error for PAM.
- ◇ Specifically, the probability of a correct decision for the M -ary QAM system is

$$P_c = \left(1 - P_{\sqrt{M}}\right)^2$$

where $p_{\sqrt{M}}$ is the probability of error of an \sqrt{M} – ary PAM.

Probability of Error for QAM



- ◇ By appropriately modifying the probability of error for M -ary PAM, we obtain

$$P_{\sqrt{M}} = 2 \left(1 - \frac{1}{\sqrt{M}} \right) Q \left(\sqrt{\frac{3}{M-1} \frac{\mathcal{E}_{av}}{N_0}} \right)$$

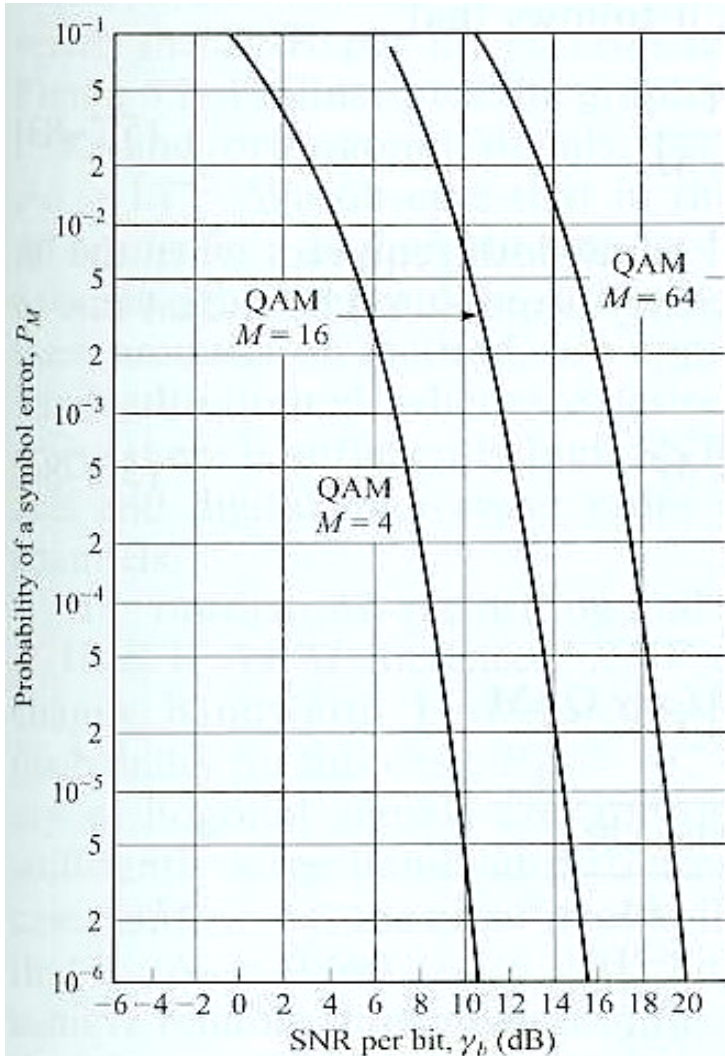
where \mathcal{E}_{av}/N_0 is the average SNR per symbol.

- ◇ Therefore, the probability of a symbol error for M -ary QAM is

$$P_M = 1 - \left(1 - P_{\sqrt{M}} \right)^2$$

- ◇ Note that this result is exact for $M = 2^k$ when k is even.
- ◇ When k is odd, there is no equivalent \sqrt{M} -ary PAM system. This is no problem, because it is rather easy to determine the error rate for a rectangular signal set.

Probability of Error for QAM



- ◇ We employ the optimum detector that bases its decisions on the optimum distance metrics, it is relatively straightforward to show that the symbol error probability is tightly upper-bounded as

$$P_M \leq 1 - \left[2Q \left(\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}} \right) \right]^2$$
$$\leq 4Q \left(\sqrt{\frac{3k\mathcal{E}_{bav}}{(M-1)N_0}} \right)$$

for any $k \geq 1$, where \mathcal{E}_{bav}/N_0 is the average SNR per bit.

Probability of Error for QAM



- ◇ For nonrectangular QAM signal constellation, we may upper-bound the error probability by use of a union bound :

$$P_M < (M - 1)Q\left(\sqrt{\left[d_{\min}^{(e)}\right]^2 / 2N_0}\right)$$

where $d_{\min}^{(e)}$ is the minimum Euclidean distance between signal points.

- ◇ This bound may be loose when M is large.
- ◇ We approximate P_M by replacing $M-1$ by M_n , where M_n is the largest number of neighboring points that are at distance $d_{\min}^{(e)}$ from any constellation point.

Comparison of Digital Modulation Methods



- ◇ One can compare the digital modulation methods on the basis of the SNR required to achieve a specified probability of error.
- ◇ However, such a comparison would not be very meaningful, unless it were made on the basis of some constraint, such as a fixed data rate of transmission or, on the basis of a fixed bandwidth.
- ◇ For multiphase signals, the channel bandwidth required is simply the bandwidth of the equivalent low-pass signal pulse $g(t)$ with duration T and bandwidth W , which is approximately equal to the reciprocal of T .
- ◇ Since $T = k/R = (\log_2 M)/R$, it follows that
$$W = \frac{R}{\log_2 M}$$

Comparison of Digital Modulation Methods



- ◇ As M is increased, the channel bandwidth required, when the bit rate R is fixed, decreases. The *bandwidth efficiency* is measured by the bit rate to bandwidth ratio, which is

$$\frac{R}{W} = \log_2 M$$

- ◇ The bandwidth-efficient method for transmitting PAM is single-sideband. The channel bandwidth required to transmit the signal is approximately equal to $1/2T$ and,

$$\frac{R}{W} = 2 \log_2 M$$

this is a factor of 2 better than PSK.

- ◇ For QAM, we have two orthogonal carriers, with each carrier having a PAM signal.

Comparison of Digital Modulation Methods



- ◇ Thus, we double the rate relative to PAM. However, the QAM signal must be transmitted via double-sideband. Consequently, QAM and PAM have the same bandwidth efficiency when the bandwidth is referenced to the band-pass signal.
- ◇ As for orthogonal signals, if the $M = 2^k$ orthogonal signals are constructed by means of orthogonal carriers with minimum frequency separation of $1/2T$, the bandwidth required for transmission of $k = \log_2 M$ information bits is

$$W = \frac{M}{2T} = \frac{M}{2(k/R)} = \frac{M}{2\log_2 M} R$$

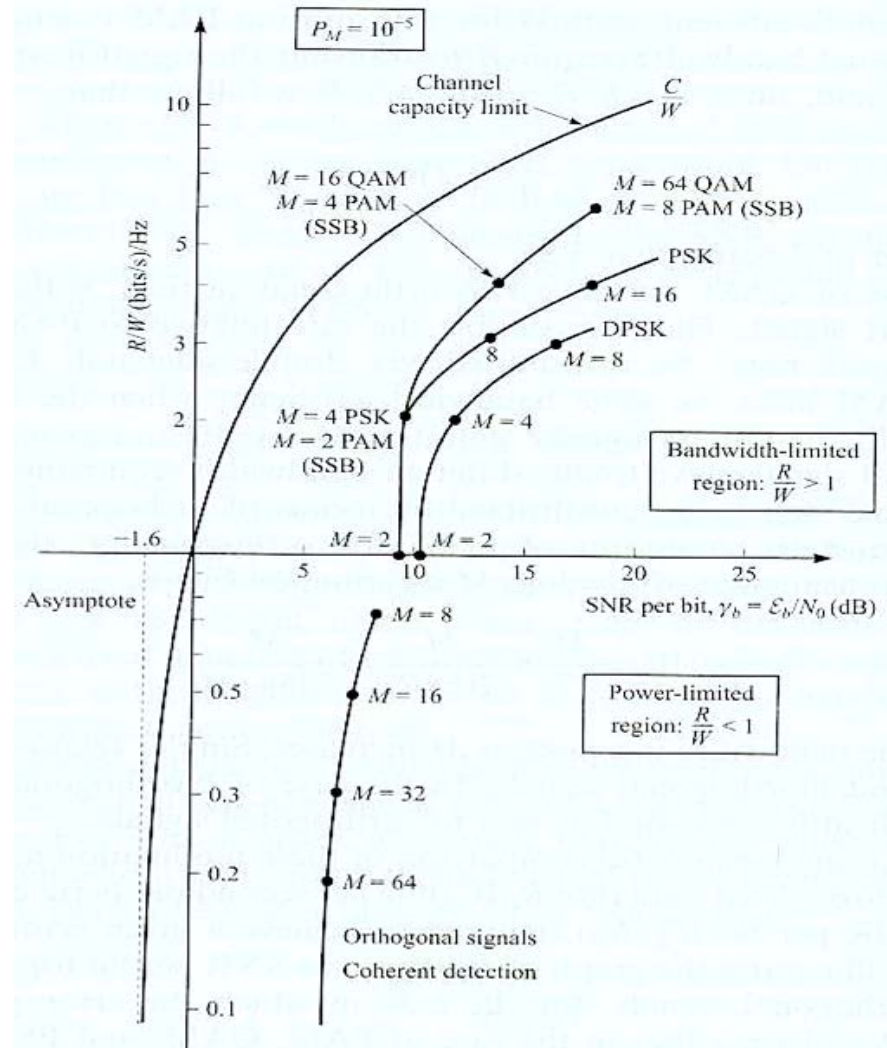
In the case, the bandwidth increases as M increases.

- ◇ In the case of biorthogonal signals, the required bandwidth is one-half of that for orthogonal signals.

Comparison of Digital Modulation Methods



- A compact and meaningful comparison of modulation methods is one based on the normalized data rate R/W (bits per second per hertz of bandwidth) versus the SNR per bit (ϵ_b/N_0) required to achieve a given error probability.
- In the case of PAM, QAM, and PSK, increasing M results in a higher bit-to-bandwidth ratio R/W .



Comparison of Digital Modulation Methods



- ◇ However, the cost of achieving the higher data rate is an increase in the SNR per bit.
- ◇ Consequently, these modulation methods are appropriate for communication channels that are bandwidth limited, where we desire a $R/W > 1$ and where there is sufficiently high SNR to support increases in M .
 - ◇ Telephone channels and digital microwave radio channels are examples of such band-limited channels.
- ◇ In contrast, M -ary orthogonal signals yield a $R/W \leq 1$. As M increases, R/W decreases due to an increase in the required channel bandwidth.
- ◇ The SNR per bit required to achieve a given error probability decreases as M increases.

Comparison of Digital Modulation Methods



- ◇ Consequently, M -ary orthogonal signals are appropriate for power-limited channels that have sufficiently large bandwidth to accommodate a large number of signals.
- ◇ As $M \rightarrow \infty$, the error probability can be made as small as desired, provided that $\text{SNR} > 0.693$ (-1.6dB). This is the minimum SNR per bit required to achieve reliable transmission in the limit as the channel bandwidth $W \rightarrow \infty$ and the corresponding $R/W \rightarrow 0$.
- ◇ The figure above also shown the normalized capacity of the band-limited AWGN channel, which is due to Shannon (1948).
- ◇ The ratio C/W , where C ($=R$) is the capacity in bits/s, represents the highest achievable bit rate-to-bandwidth ratio on this channel.
- ◇ Hence, it serves the upper bound on the bandwidth efficiency of any type of modulation.

Optimum Receiver for Signals with Random Phase in AWGN Channel



- ◇ In this section, we consider the design of the optimum receiver for carrier modulated signals when the carrier phase is unknown and no attempt is made to estimate its value.
- ◇ Uncertainty in the carrier phase of the receiver signal may be due to one or more of the following reasons:
 - ◇ The oscillators that are used at the transmitter and the receiver to generate the carrier signals are generally not phase synchronous.
 - ◇ The time delay in the propagation of the signal from the transmitter to the receiver is not generally known precisely.
- ◇ Assuming a transmitted signal of the form

$$s(t) = \text{Re} \left[g(t) e^{j2\pi f_c t} \right]$$

that propagates through a channel with delay t_0 will be received as:

$$s(t - t_0) = \text{Re} \left[g(t - t_0) e^{j2\pi f_c (t - t_0)} \right] = \text{Re} \left[g(t - t_0) e^{-j2\pi f_c t_0} e^{j2\pi f_c t} \right]$$

Optimum Receiver for Signals with Random Phase in AWGN Channel



- ◇ The carrier phase shift due to the propagation delay t_0 is
$$\phi = -2\pi f_c t_0$$
- ◇ Note that large changes in the carrier phase can occur due to relatively small changes in the propagation delay.
- ◇ For example, if the carrier frequency $f_c = 1$ MHz, an uncertainty or a change in the propagation delay of $0.5\mu\text{s}$ will cause a phase uncertainty of π rad.
- ◇ In some channels the time delay in the propagation of the signal from the transmitter to the receiver may change rapidly and in an apparently random fashion.
- ◇ In the absence of the knowledge of the carrier phase, we may treat this signal parameter as a random variable and determine the form of the optimum receiver for recovering the transmitted information from the received signal.

Optimum Receiver for Binary Signals



- ◇ We consider a binary communication system that uses the two carrier modulated signal $s_1(t)$ and $s_2(t)$ to transmit the information, where:

$$s_m(t) = \text{Re} \left[s_{lm}(t) e^{j2\pi f_c t} \right], \quad m = 1, 2, \quad 0 \leq t \leq T$$

and $s_{lm}(t)$, $m=1,2$ are the equivalent low-pass signals.

- ◇ The two signals are assumed to have equal energy

$$\mathcal{E} = \int_0^T s_m^2(t) dt = \frac{1}{2} \int_0^T |s_{lm}(t)|^2 dt$$

- ◇ The two signals are characterized by the complex-valued correlation coefficient

$$\rho_{12} \equiv \rho = \frac{1}{2\mathcal{E}} \int_0^T s_{l1}^*(t) s_{l2}(t) dt$$

Optimum Receiver for Binary Signals



- ◇ The received signal is assumed to be a phase-shifted version of the transmitted signal and corrupted by the additive noise

$$n(t) = \text{Re} \left\{ \left[n_c(t) + jn_s(t) \right] \cdot e^{j2\pi f_c t} \right\} = \text{Re} \left[z(t) \cdot e^{j2\pi f_c t} \right]$$

- ◇ The received signal may be expressed as

$$r(t) = \text{Re} \left\{ \left[s_{lm}(t) e^{j\phi} + z(t) \right] \cdot e^{j2\pi f_c t} \right\}$$

where $r_l(t) = s_{lm}(t) e^{j\phi} + z(t)$, $0 \leq t \leq T$

$r_l(t)$ is the equivalent low-pass received signal.

- ◇ This received signal is now passed through a demodulator whose sampled output at $t = T$ is passed to the detector.

Optimum Receiver for Binary Signals



- ◇ The optimum demodulator
 - ◇ We have demonstrated that if the received signals was correlated with a set of orthogonal functions $\{f_n(t)\}$ that spanned the signal space, the outputs from the bank of correlators provide a set of sufficient statistics for the detector to make a decision that minimizes the probability error.
 - ◇ We also demonstrated that a bank of matched filters could be substituted for the bank of correlations.
 - ◇ A similar orthogonal decomposition can be employed for a received signal with an unknown carrier phase.
 - ◇ It is mathematically convenient to deal with the equivalent low-pass signal and to specify the signal correlators or matched filters in terms of the equivalent low-pass signal waveforms.

Optimum Receiver for Binary Signals

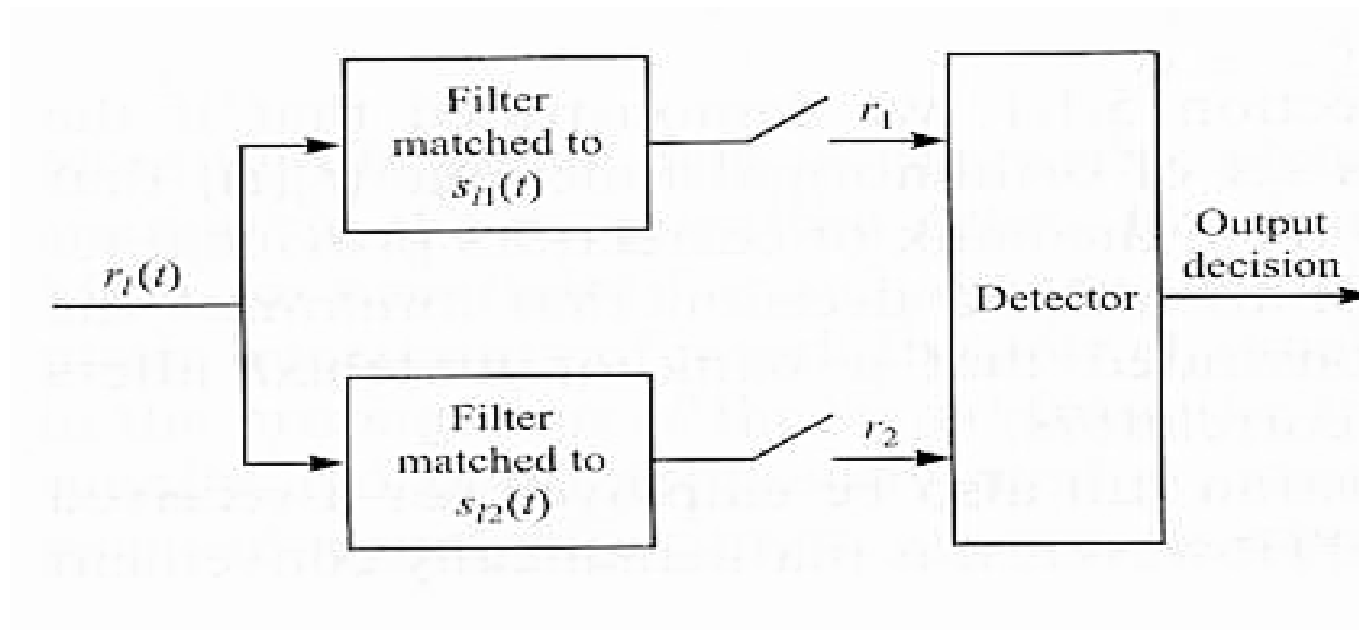


- ◇ The optimum demodulator
 - ◇ The impulse response $h_l(t)$ of a filter that is matched to the complex-valued equivalent low-pass signal $s_l(t)$, $0 \leq t \leq T$ is given as $h_l(t) = s_l^*(T-t)$ and the output of such filter at $t=T$ is simply $\int_0^T |s_l(t)|^2 dt = 2\varepsilon$ where ε is the signal energy.
 - ◇ A similar result is obtained if the signal $s_l(t)$ is correlated with $s_l^*(t)$ and the correlator is sampled signal $s_l(t)$ at $t=T$.
 - ◇ The optimum demodulator for the equivalent low-pass received signal $s_l(t)$ may be realized by two matched filters in parallel, one matched to $s_{l1}(t)$ and the other to $s_{l2}(t)$.

Optimum Receiver for Binary Signals



- ◇ The optimum demodulator
 - ◇ Optimum receiver for binary signals



- ◇ The output of the matched filters or correlators at the sampling instant are the two complex numbers

$$r_m = r_{mc} + jr_{ms}, \quad m = 1,2$$

Optimum Receiver for Binary Signals



- ◇ The optimum demodulator

- ◇ Suppose that the transmitted signal is $s_1(t)$. It can be shown that

$$r_1 = 2\varepsilon \cos \phi + n_{1c} + j(2\varepsilon \sin \phi + n_{1s}).$$

$$r_2 = 2\varepsilon|\rho| \cos(\phi - \alpha_0) + n_{2c} + j[2\varepsilon|\rho| \sin(\phi - \alpha_0) + n_{2s}]$$

where ρ is the complex-valued correlation coefficient of two signals $s_{11}(t)$ and $s_{12}(t)$, which may be expressed as $\rho = |\rho| \exp(j\alpha_0)$.

- ◇ The random noise variables n_{1c} , n_{1s} , n_{2c} , and n_{2s} are jointly Gaussian, with zero-mean and equal variance.

Optimum Receiver for Binary Signals



◇ The optimum detector

- ◇ The optimum detector observes the random variables $[r_{1c} \ r_{1s} \ r_{2c} \ r_{2s}] = \mathbf{r}$, where $r_1 = r_{1c} + jr_{1s}$, and $r_2 = r_{2c} + jr_{2s}$, and bases its decision on the posterior probabilities $P(\mathbf{s}_m | \mathbf{r})$, $m=1,2$.
- ◇ These probabilities may be expressed as

$$P(\mathbf{s}_m | \mathbf{r}) = \frac{p(\mathbf{r} | \mathbf{s}_m)P(\mathbf{s}_m)}{p(\mathbf{r})}, \quad m = 1,2$$

- ◇ The optimum decision rule may be expressed as

$$P(\mathbf{s}_1 | \mathbf{r}) \underset{s_2}{\overset{s_1}{\gtrless}} P(\mathbf{s}_2 | \mathbf{r}) \quad \text{or} \quad \frac{p(\mathbf{r} | \mathbf{s}_1)}{p(\mathbf{r} | \mathbf{s}_2)} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{P(\mathbf{s}_2)}{P(\mathbf{s}_1)}$$

▶ Likelihood ratio.

Optimum Receiver for Binary Signals



- ◇ The optimum detector

- ◇ The ratio of PDFs on the left-hand side is the *likelihood ratio*, which we denote as

$$\Lambda(\mathbf{r}) = \frac{p(\mathbf{r} | \mathbf{s}_1)}{p(\mathbf{r} | \mathbf{s}_2)}$$

- ◇ The right-hand side is the ratio of the two prior probabilities, which takes the value of unity when the two signals are equally probable.
- ◇ The probability density functions $p(\mathbf{r}|\mathbf{s}_1)$ and $p(\mathbf{r}|\mathbf{s}_2)$ can be obtained by averaging the PDFs $p(\mathbf{r}|\mathbf{s}_m, \phi)$ over the PDF of the random carrier phase, i.e.,

$$p(\mathbf{r} | \mathbf{s}_m) = \int_0^{2\pi} p(\mathbf{r} | \mathbf{s}_m, \phi) p(\phi) d\phi$$

Optimum Receiver for Binary Signals



- ◇ The optimum detector
 - ◇ For the special case in which the two signals are orthogonal, i.e., $\rho=0$, the outputs of the demodulator are:

$$\begin{aligned}r_1 &= r_{1c} + jr_{1s} \\ &= 2\varepsilon \cos \phi + n_{1c} + j(2\varepsilon \sin \phi + n_{1s})\end{aligned}$$

$$\begin{aligned}r_2 &= r_{2c} + jr_{2s} \\ &= n_{2c} + jn_{2s}\end{aligned}$$

where $(n_{1c}, n_{1s}, n_{2c}, n_{2s})$ are mutually uncorrelated and, statistically independent, zero-mean Gaussian random variable.

Optimum Receiver for Binary Signals



◇ The optimum detector

◇ The joint PDF of $\mathbf{r}=[r_{1c} \ r_{1s} \ r_{2c} \ r_{2s}]$ may be expressed as a product of the marginal PDFs.

◇ Consequently

$$p(r_{1c}, r_{1s} | \mathbf{s}_1, \phi) = \frac{1}{2\pi\sigma^2} \exp \left[-\frac{(r_{1c} - 2\varepsilon \cos \phi)^2 + (r_{1s} - 2\varepsilon \sin \phi)^2}{2\sigma^2} \right]$$

$$p(r_{2c}, r_{2s} | \mathbf{s}_1, \phi) = \frac{1}{2\pi\sigma^2} \exp \left(-\frac{r_{2c}^2 + r_{2s}^2}{2\sigma^2} \right)$$

where $\sigma^2=2\varepsilon N_0$.

◇ The uniform PDF for the carrier phase ϕ ($p(\phi)=1/2\pi$) represents the most ignorance that can be exhibited by the detector.

◇ This is called the *least favorable PDF* for ϕ .

Optimum Receiver for Binary Signals



◇ The optimum detector

- ◇ With $p(\phi)=1/2\pi$, $0\leq\phi\leq2\pi$, substituted into the integral in $p(\mathbf{r}|\mathbf{s}_m)$, we obtain:

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} p(r_{1c}, r_{1s} | \mathbf{s}_1, \phi) d\phi \\ &= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r_{1c}^2 + r_{1s}^2 + 4\varepsilon^2}{2\sigma^2}\right) \frac{1}{2\pi} \int_0^{2\pi} \exp\left[\frac{2\varepsilon(r_{1c} \cos \phi + r_{1s} \sin \phi)}{\sigma^2}\right] d\phi \end{aligned}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \exp\left[\frac{2\varepsilon(r_{1c} \cos \phi + r_{1s} \sin \phi)}{\sigma^2}\right] d\phi = I_0\left(\frac{2\varepsilon\sqrt{r_{1c}^2 + r_{1s}^2}}{\sigma^2}\right)$$

where $I_0(x)$ is the modified Bessel function of zeroth order.

Optimum Receiver for Binary Signals



◇ The optimum detector

- ◇ By performing a similar integration under the assumption that the signal $s_2(t)$ was transmitted, we obtain the result

$$p(r_{2c}, r_{2s} | \mathbf{s}_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r_{2c}^2 + r_{2s}^2 + 4\varepsilon^2}{2\sigma^2}\right) I_0\left(\frac{2\varepsilon\sqrt{r_{2c}^2 + r_{2s}^2}}{\sigma^2}\right)$$

- ◇ When we substitute these results into the likelihood ratio given by equation $\Lambda(\mathbf{r})$, we obtain the result (see P.134)

$$\Lambda(\mathbf{r}) = \frac{p(\mathbf{r} | \mathbf{s}_1)}{p(\mathbf{r} | \mathbf{s}_2)} = \frac{P(r_{1c}r_{1s} | s_1)P(r_{2c}r_{2s} | s_1)}{P(r_{1c}r_{1s} | s_2)P(r_{2c}r_{2s} | s_2)} = \frac{I_0\left(2\varepsilon\sqrt{r_{1c}^2 + r_{1s}^2}/\sigma^2\right)}{I_0\left(2\varepsilon\sqrt{r_{2c}^2 + r_{2s}^2}/\sigma^2\right)} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{P(\mathbf{s}_2)}{P(\mathbf{s}_1)}$$

- ◇ The optimum detector computes the two envelopes $\sqrt{r_{1c}^2 + r_{1s}^2}$ and $\sqrt{r_{2c}^2 + r_{2s}^2}$ and the corresponding values of the Bessel function $I_0\left(2\varepsilon\sqrt{r_{1c}^2 + r_{1s}^2}/\sigma^2\right)$ and $I_0\left(2\varepsilon\sqrt{r_{2c}^2 + r_{2s}^2}/\sigma^2\right)$ to form the likelihood ratio.

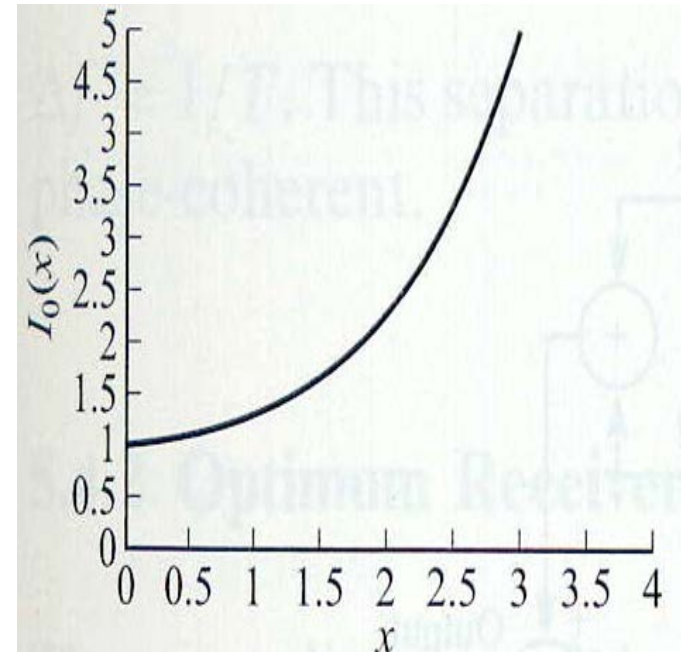
Optimum Receiver for Binary Signals



- ◇ The optimum detector
 - ◇ we observe that this computation requires knowledge of the noise variance σ^2 .
 - ◇ The likelihood ratio is then compared with the threshold $P(s_2)/P(s_1)$ to determine which signal was transmitted.

- ◇ A significant simplification in the implementation of the optimum detector occurs when the two signals are equally probable. In such a case the threshold becomes unity, and, due to the monotonicity of Bessel function shown in figure, the optimum detection rule simplifies to

$$\sqrt{r_{1c}^2 + r_{1s}^2} \underset{s_2}{\overset{s_1}{\gtrless}} \sqrt{r_{2c}^2 + r_{2s}^2}$$



Optimum Receiver for Binary Signals



- ◇ The optimum detector
 - ◇ The optimum detector bases its decision on the two envelopes and $\sqrt{r_{1c}^2 + r_{1s}^2}$, and $\sqrt{r_{2c}^2 + r_{2s}^2}$, it is called an *envelope detector*.
 - ◇ We observe that the computation of the envelopes of the received signal samples at the output of the demodulator renders the carrier phase irrelevant in the decision as to which signal was transmitted.
 - ◇ Equivalently, the decision may be based on the computation of the squared envelope $r_{1c}^2 + r_{1s}^2$ and $r_{2c}^2 + r_{2s}^2$, in which case the detector is called a *square-law detector*.

Optimum Receiver for Binary Signals



◇ Detection of binary FSK signal

- ◇ Recall that in binary FSK we employ two different frequencies, say f_1 and $f_2=f_1+\Delta f$, to transmit a binary information sequence.
- ◇ The choice of minimum frequency separation $\Delta f = f_2 - f_1$ is considered below:

$$s_1(t) = \sqrt{2\varepsilon_b/T_b} \cos 2\pi f_1 t, \quad s_2(t) = \sqrt{2\varepsilon_b/T_b} \cos 2\pi f_2 t, \quad 0 \leq t \leq T_b$$

- ◇ The equivalent low-pass counterparts are:

$$s_{l1}(t) = \sqrt{2\varepsilon_b/T_b}, \quad s_{l2}(t) = \sqrt{2\varepsilon_b/T_b} e^{j2\pi\Delta f t}, \quad 0 \leq t \leq T_b$$

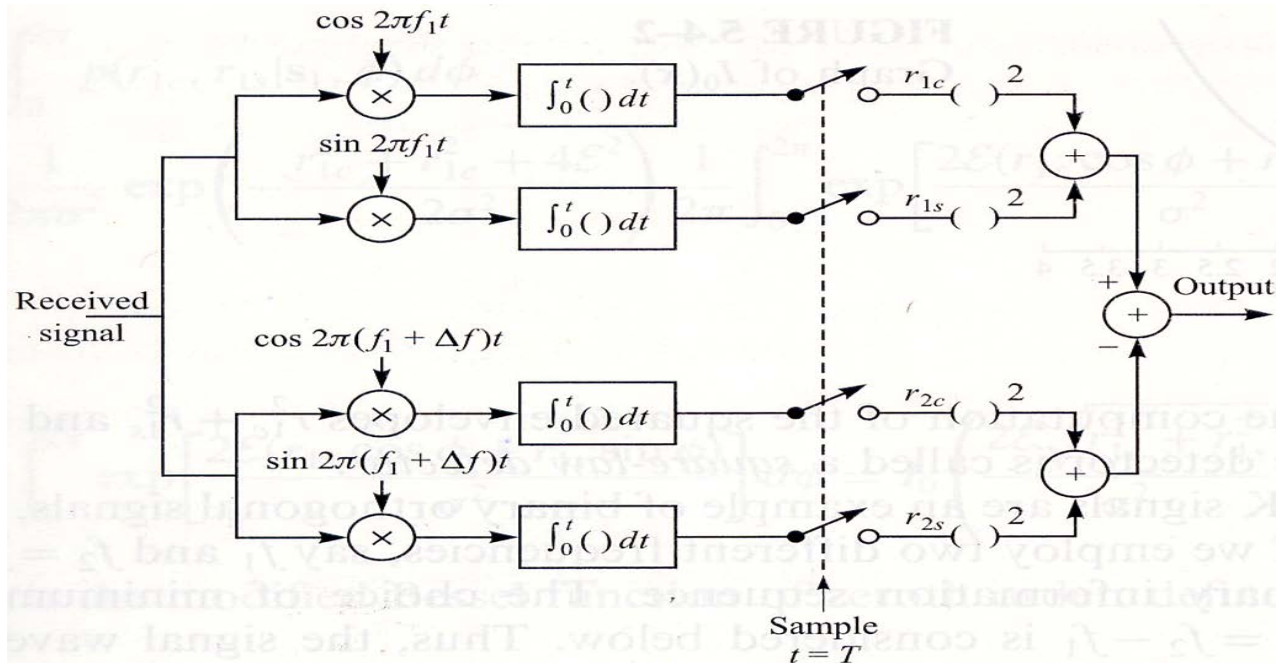
- ◇ The received signal may be expressed as:

$$r(t) = \sqrt{\frac{2\varepsilon_b}{T_b}} \cos(2\pi f_m t + \phi_m) + n(t), \quad m = 0, 1$$

Optimum Receiver for Binary Signals



- ◇ Detection of binary FSK signal
 - ◇ Demodulation and square-law detection:



$$f_{kc}(t) = \sqrt{\frac{2}{T_b}} \cos \left[(2\pi f_1 + 2\pi k \Delta f) t \right], \quad k = 0, 1$$

$$f_{ks}(t) = \sqrt{\frac{2}{T_b}} \sin \left[(2\pi f_1 + 2\pi k \Delta f) t \right], \quad k = 0, 1$$

Optimum Receiver for Binary Signals



◇ Detection of binary FSK signal

- ◇ If the m th signal is transmitted, the four samples at the detector may be expressed as:

$$\begin{aligned}
 r_{kc} &= \int_0^{T_b} r(t) \cdot f_{kc}(t) dt = \int_0^{T_b} \left\{ \sqrt{\frac{2\varepsilon_b}{T_b}} \cos[2\pi(f_1 + m\Delta f)t + \phi_m] + n(t) \right\} \cdot \left\{ \sqrt{\frac{2}{T_b}} \cos[(2\pi f_1 + 2\pi k\Delta f)t] \right\} dt \\
 &= \frac{2\sqrt{\varepsilon_b}}{T_b} \int_0^{T_b} \left\{ \cos[2\pi(f_1 + m\Delta f)t] \cos \phi_m - \sin[2\pi(f_1 + m\Delta f)t] \sin \phi_m \right\} \cdot \left\{ \cos[2\pi(f_1 + k\Delta f)t] \right\} dt + n_{kc} \\
 &= \frac{\sqrt{\varepsilon_b}}{T_b} \int_0^{T_b} \left\{ \cos[2\pi(m-k)\Delta ft] + \cos[2\pi(2f_1 + (m+k)\Delta f)t] \right\} \cos \phi_m - \\
 &\quad \left\{ \sin[2\pi(m-k)\Delta ft] - \sin[2\pi(2f_1 + (m+k)\Delta f)t] \right\} \sin \phi_m dt + n_{kc} \\
 &= \frac{\sqrt{\varepsilon_b}}{T_b} \left\{ \frac{\sin[2\pi(m-k)\Delta ft]}{2\pi(m-k)\Delta f} \cos \phi_m + \frac{\cos[2\pi(m-k)\Delta ft]}{2\pi(m-k)\Delta f} \sin \phi_m \right\}_0^{T_b} + n_{kc} \\
 &= \sqrt{\varepsilon_b} \left\{ \frac{\sin[2\pi(m-k)\Delta f T_b]}{2\pi(m-k)\Delta f T_b} \cos \phi_m + \frac{\cos[2\pi(m-k)\Delta f T_b] - 1}{2\pi(m-k)\Delta f T_b} \sin \phi_m \right\} + n_{kc} \quad k, m = 0, 1
 \end{aligned}$$

Optimum Receiver for Binary Signals



◇ Detection of binary FSK signal

- ◇ If the m th signal is transmitted, the four samples at the detector may be expressed as (cont.):

$$\begin{aligned}
 r_{ks} &= \int_0^{T_b} r(t) \cdot f_{ks}(t) dt = \int_0^{T_b} \left\{ \sqrt{\frac{2\varepsilon_b}{T_b}} \cos[2\pi(f_1 + m\Delta f)t + \phi_m] + n(t) \right\} \cdot \left\{ \sqrt{\frac{2}{T_b}} \sin[(2\pi f_1 + 2\pi k\Delta f)t] \right\} dt \\
 &= \frac{2\sqrt{\varepsilon_b}}{T_b} \int_0^{T_b} \left\{ \cos[2\pi(f_1 + m\Delta f)t] \cos \phi_m - \sin[2\pi(f_1 + m\Delta f)t] \sin \phi_m \right\} \cdot \left\{ \sin[2\pi(f_1 + k\Delta f)t] \right\} dt + n_{ks} \\
 &= \frac{\sqrt{\varepsilon_b}}{T_b} \int_0^{T_b} \left\{ -\sin[2\pi(m-k)\Delta ft] + \sin[2\pi(2f_1 + (m+k)\Delta f)t] \right\} \cos \phi_m - \\
 &\quad \left\{ \cos[2\pi(m-k)\Delta ft] - \cos[2\pi(2f_1 + (m+k)\Delta f)t] \right\} \sin \phi_m dt + n_{ks} \\
 &= \frac{\sqrt{\varepsilon_b}}{T_b} \left\{ \frac{\cos[2\pi(m-k)\Delta ft]}{2\pi(m-k)\Delta f} \cos \phi_m - \frac{\sin[2\pi(m-k)\Delta ft]}{2\pi(m-k)\Delta f} \sin \phi_m \right\}_0^{T_b} + n_{ks} \\
 &= \sqrt{\varepsilon_b} \left\{ \frac{\cos[2\pi(m-k)\Delta f T_b] - 1}{2\pi(m-k)\Delta f T_b} \cos \phi_m - \frac{\sin[2\pi(m-k)\Delta f T_b]}{2\pi(m-k)\Delta f T_b} \sin \phi_m \right\} + n_{ks} \quad k, m = 0, 1
 \end{aligned}$$

Optimum Receiver for Binary Signals



◇ Detection of binary FSK signal

- ◇ We observe that when $k=m$, the sampled values to the detector are

$$\begin{aligned} r_{mc} &= \sqrt{\varepsilon_b} \cos \phi_m + n_{mc} \\ r_{ms} &= -\sqrt{\varepsilon_b} \sin \phi_m + n_{ms} \end{aligned} \quad k=m$$

- ◇ We observe that when $k \neq m$, the signal components in the samples r_{kc} and r_{ks} will vanish, independently of the values of the phase shifts ϕ_k , provided that the frequency separation between successive frequency is $\Delta f = 1/T$.
- ◇ In such case, the other two correlator outputs consist of noise only, i.e.,

$$r_{kc} = n_{kc}, \quad r_{ks} = n_{ks}, \quad k \neq m.$$

Optimum Receiver for M -ary Orthogonal Signals



- ◇ If the equal energy and equally probable signal waveforms are represented as

$$s_m(t) = \text{Re} \left[s_{lm}(t) e^{j2\pi f_c t} \right], \quad m = 1, 2, \dots, M, \quad 0 \leq t \leq T$$

where $s_{lm}(t)$ are the equivalent low-pass signals.

- ◇ The optimum correlation-type or matched-filter-type demodulator produces the M complex-valued random variables

$$\begin{aligned} r_m &= r_{mc} + jr_{ms} = \int_0^T r_l(t) h_{lm}^*(T-t) dt \\ &= \int_0^T r_l(t) s_{lm}^*(T-(T-t)) dt = \int_0^T r_l(t) s_{lm}^*(t) dt, \quad m = 1, 2, \dots, M \end{aligned}$$

where $r_l(t)$ is the equivalent low-pass signals.

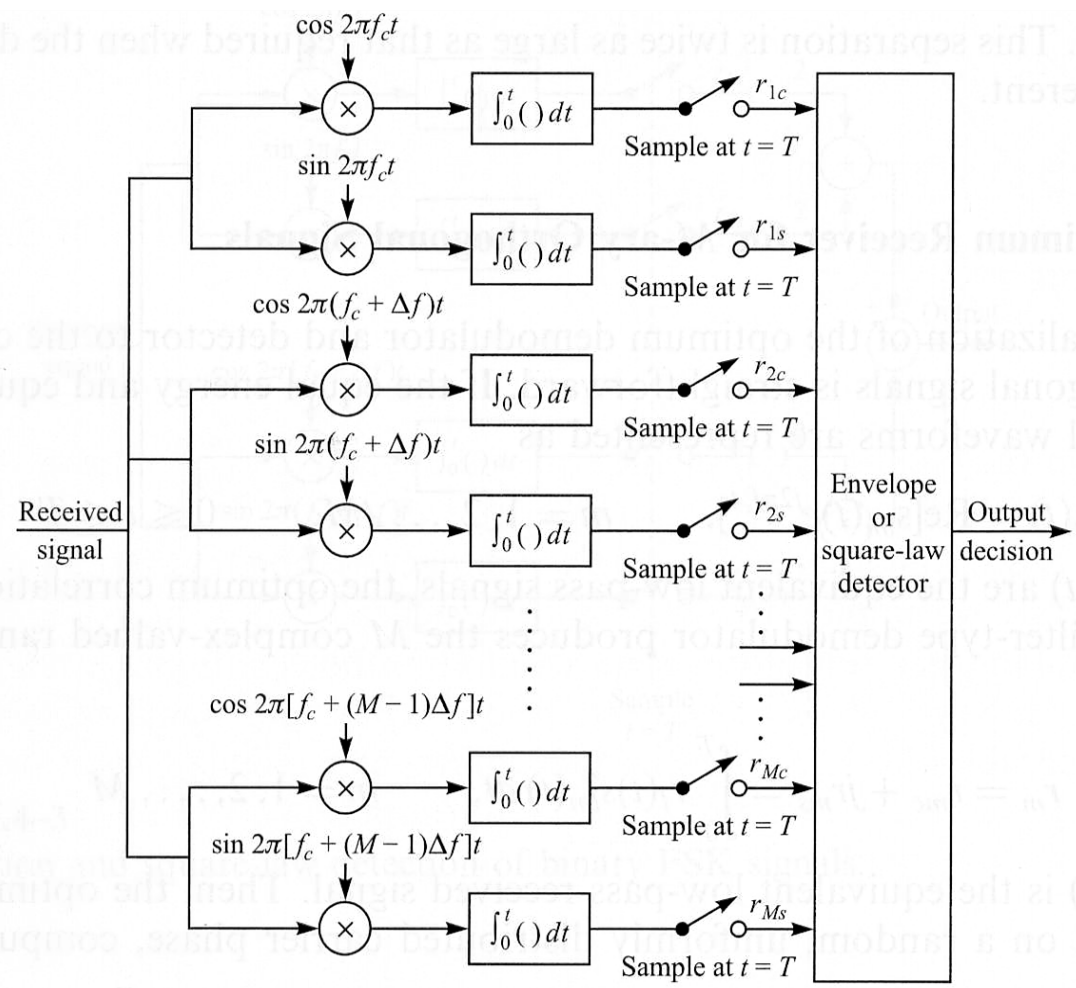
- ◇ The optimum detector, based on a random, uniformly distributed carrier phase, computes the M envelopes

$$|r_m| = \sqrt{r_{mc}^2 + r_{ms}^2}, \quad m = 1, 2, \dots, M$$

or, the squared envelopes $|r_m|^2$, and selects the signal with the largest envelope.

Optimum Receiver for M -ary Orthogonal Signals

- ◇ Optimum receiver for M -ary orthogonal FSK signals.
 - ◇ There are $2M$ correlators: two for each possible transmitted frequency.
 - ◇ The minimum frequency separation between adjacent frequencies to maintain orthogonality is $\Delta f = 1/T$.



Demodulation of M -ary signals for noncoherent detection

Probability of Error for Envelope Detection of M -ary Orthogonal Signals

- ◇ We assume that the M signals are equally probable a priori and that the signal $s_1(t)$ is transmitted in the signal interval $0 \leq t \leq T$.
- ◇ The M decision metrics at the detector are the M envelopes

$$|r_m| = \sqrt{r_{mc}^2 + r_{ms}^2}, \quad m = 1, 2, \dots, M$$

where $r_{1c} = \sqrt{\varepsilon_s} \cos \phi_1 + n_{1c}$

$$r_{1s} = \sqrt{\varepsilon_s} \sin \phi_1 + n_{1s}$$

and $r_{mc} = n_{mc}$, $r_{ms} = n_{ms}$, $m = 2, 3, \dots, M$

- ◇ The additive noise components $\{n_{mc}\}$ and $\{n_{ms}\}$ are mutually statistically independent zero-mean Gaussian variables with equal variance $\sigma^2 = N_0/2$.

Probability of Error for Envelope Detection of M -ary Orthogonal Signals



- ◇ The PDFs of the random variables at the input to the detector are

$$p_{r_1}(r_{1c}, r_{1s}) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r_{1c}^2 + r_{1s}^2 + \varepsilon_s}{2\sigma^2}\right) I_0\left(\frac{\sqrt{\varepsilon_s (r_{1c}^2 + r_{1s}^2)}}{\sigma^2}\right)$$

$$p_{r_m}(r_{mc}, r_{ms}) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r_{mc}^2 + r_{ms}^2}{2\sigma^2}\right), \quad m = 2, 3, \dots, M$$

- ◇ We define the normalized variables

$$R_m = \frac{\sqrt{r_{mc}^2 + r_{ms}^2}}{\sigma}$$

$$\Theta_m = \tan^{-1} \frac{r_{ms}}{r_{mc}}$$

Probability of Error for Envelope Detection of M -ary Orthogonal Signals

- Clearly, $r_{mc} = \sigma R_m \cos \Theta_m$ and $r_{ms} = \sigma R_m \sin \Theta_m$. The Jacobian of this transformation is

$$|J| = \begin{vmatrix} \sigma \cos \Theta_m & \sigma \sin \Theta_m \\ -\sigma R_m \sin \Theta_m & \sigma R_m \cos \Theta_m \end{vmatrix} = \sigma^2 R_m$$

- Consequently,

$$p(R_1, \Theta_1) = \frac{R_1}{2\pi} \exp \left[-\frac{1}{2} \left(R_1^2 + 2 \frac{\varepsilon_s}{N_0} \right) \right] I_0 \left(\sqrt{\frac{2\varepsilon_s}{N_0}} R_1 \right)$$

$$p(R_m, \Theta_m) = \frac{R_m}{2\pi} \exp \left(-\frac{1}{2} R_m^2 \right), \quad m = 2, 3, \dots, M$$

- Finally, by averaging $p(R_m, \Theta_m)$ over Θ_m , the factor of 2π is eliminated.
- Thus, we find that R_1 has a Rice probability distribution and R_m , $m=2, 3, \dots, M$, are each Rayleigh-distributed.

Probability of Error for Envelope Detection of M -ary Orthogonal Signals

- ◇ The probability of a correct decision is simply the probability that $R_1 > R_2$, and $R_1 > R_3, \dots$, and $R_1 > R_m$. Hence,

$$\begin{aligned}
 P_c &= P(R_2 < R_1, R_3 < R_1, \dots, R_M < R_1) \\
 &= \int_0^\infty P(R_2 < R_1, R_3 < R_1, \dots, R_M < R_1 \mid R_1 = x) p_{R_1}(x) dx
 \end{aligned}$$

- ◇ Because the random variables R_m , $m=2,3,\dots,M$, are statistically independent and identically distributed, the joint probability conditioned on R_1 factors into a product of $M-1$ identical terms.

where
$$P_c = \int_0^\infty \left[P(R_2 < R_1 \mid R_1 = x) \right]^{M-1} p_{R_1}(x) dx \quad (*)$$

$$P(R_2 < R_1 \mid R_1 = x) = \int_0^x p_{R_2}(r_2) dr_2 = 1 - e^{-x^2/2}$$

Probability of Error for Envelope Detection of M -ary Orthogonal Signals



- ◇ The $(M-1)$ th power may be expressed as

$$\left(1 - e^{-x^2/2}\right)^{M-1} = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} e^{-nx^2/2}$$

- ◇ Substitution of this result into Equation (*) in Page 150 and integration over x yield the probability of a correct decision as

$$P_c = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \frac{1}{n+1} \exp\left[\frac{n\varepsilon_s}{(n+1)N_0}\right]$$

where ε_s/N_0 is the SNR per symbol.

- ◇ The probability of a symbol error, which is $P_m = 1 - P_c$, becomes

$$P_M = \sum_{n=1}^{M-1} (-1)^{n+1} \binom{M-1}{n} \frac{1}{n+1} \exp\left[-\frac{nk\varepsilon_b}{(n+1)N_0}\right]$$

where ε_b/N_0 is the SNR per bit.

Probability of Error for Envelope Detection of M -ary Orthogonal Signals



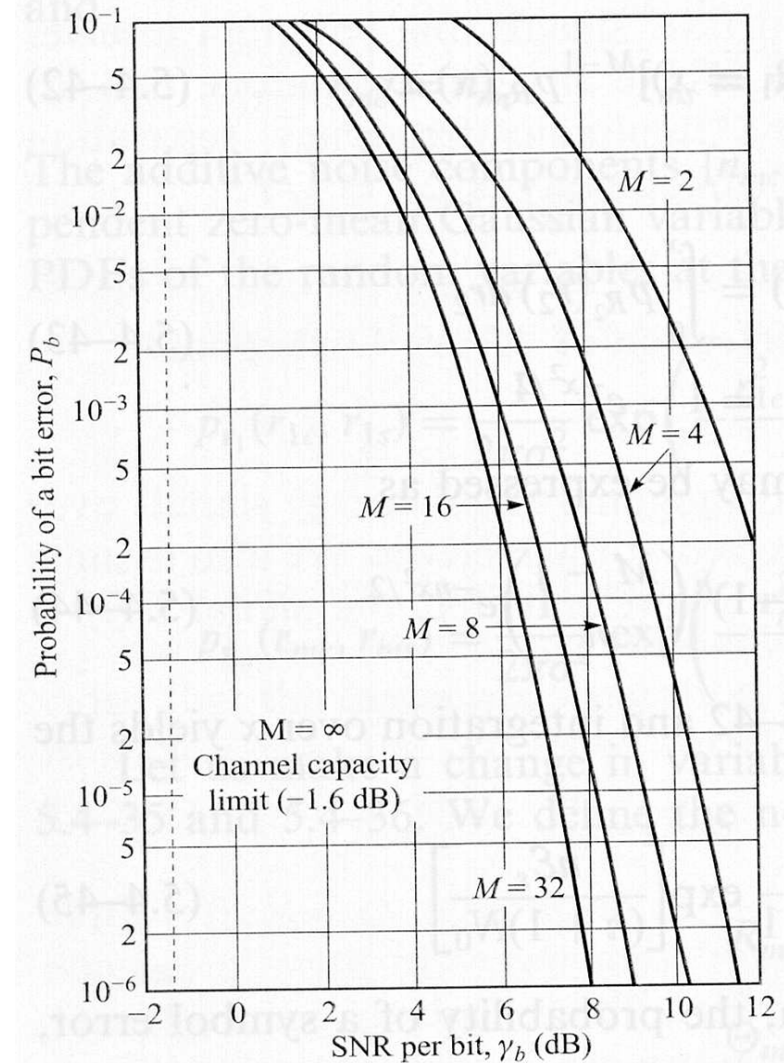
- For binary orthogonal signals ($M=2$), Equation reduces to the simple form

$$P_2 = \frac{1}{2} e^{-\varepsilon_b/2N_0}$$

- For $M>2$, we may compute the probability of a bit error by making use of the relationship

$$P_b = \frac{2^{k-1}}{2^k - 1} P_M$$

- Figure shows the bit-error probability as a function of SNR per bit γ_b for $M=2,4,8,16$, and 32.



Probability of Error for Envelope Detection of M -ary Orthogonal Signals



- ◇ Just as in the case of coherent detection of M -ary orthogonal signals, we observe that for any given bit-error probability, the SNR per bit decreases as M increase.
- ◇ It can be shown that, in the limit as $M \rightarrow \infty$, the probability of bit error P_b can be made arbitrarily small provided that the SNR per bit is greater than the Shannon limit of -1.6dB.
- ◇ The cost for increasing M is the bandwidth required to transmit the signals.
- ◇ For M -ary FSK, the frequency separation between adjacent frequencies is $\Delta f = 1/T$ for signal orthogonality.
- ◇ The bandwidth required for the M signals is $W = M \Delta f = M/T$.

Probability of Error for Envelope Detection of M -ary Orthogonal Signals



- ◇ The bit rate is $R=k/T$, where $k=\log_2 M$.
- ◇ Therefore, the bit rate-to-bandwidth ratio is

$$\frac{R}{W} = \frac{\log_2 M}{M}$$