# Digital Modulation Schemes

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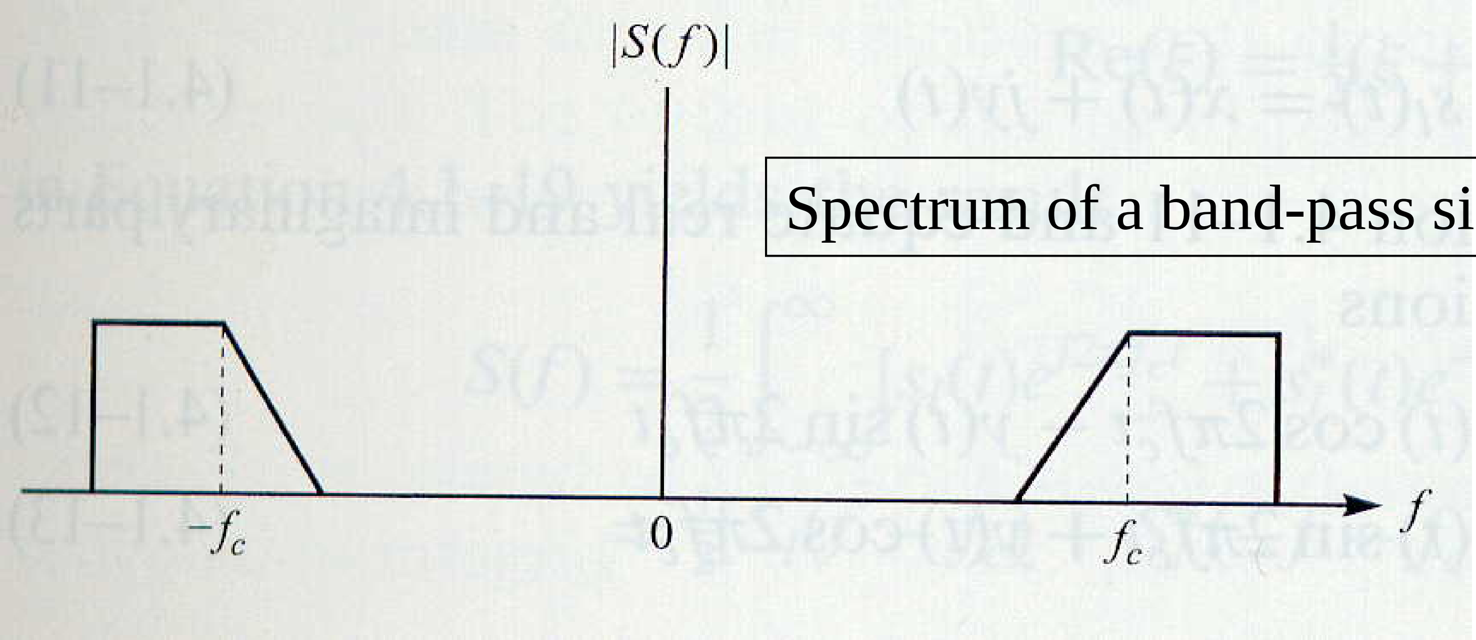
# Representation of Band-Pass Signals and Systems

- The channel over which the signal is transmitted is limited in bandwidth to an interval of frequencies centered about the carrier.
- Signals and channels (systems) that satisfy the condition that their bandwidth is much smaller than the carrier frequency are termed *narrowband band-pass signals and channels (systems)*.
- With no loss of generality and for mathematical convenience, it is desirable to reduce all band-pass signals and channels to equivalent low-pass signals and channels.

# Representation of Band-Pass Signals and Systems

- Suppose that a real-valued signal $s(t)$ has a frequency content concentrated in a narrow band of frequencies in the vicinity of a frequency $f_c$, as shown in the following figure:



Spectrum of a band-pass signal.

Our object is to develop a mathematical representation of such signals.

# Representation of Band-Pass Signals and Systems

- A signal that contains only the positive frequencies in $s(t)$ may be expressed as:

$$S_+(f) = 2u(f)S(f)$$

$$s_+(t) = \int_{-\infty}^{\infty} S_+(f) \cdot e^{j2\pi ft} df$$

$$= F^{-1}\left[2u(f)\right] * F^{-1}\left[S(f)\right]$$

where $S(f)$ is the Fourier transform of $s(t)$ and $u(f)$ is the unit step function, and the signal $s_+(t)$ is called the *analytic signal* or the *pre-envelope* of $s(t)$.

$$F^{-1}\left[2u(f)\right] = \delta(t) + \frac{j}{\pi t}$$

$$s_+(t) = \left[\delta(t) + \frac{j}{\pi t}\right] * s(t) = s(t) + j\frac{1}{\pi t} * s(t) \equiv s(t) + j\hat{s}(t)$$

# Representation of Band-Pass Signals and Systems

- Define:

$$\hat{s}(t)=\frac{1}{\pi t}*s(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}d\tau$$

Real-Valued

- A filter, called a *Hilbert transformer*, is defined as:

$$h(t)=\frac{1}{\pi t},\qquad -\infty<t<\infty$$

- The signal $\hat{s}(t)$ may be viewed as the output of the Hilbert transformer when excited by the input signal $s(t)$.
- The frequency response of this filter is:

$$H(f)=\int_{-\infty}^{\infty}h(t)e^{-j2\pi ft}dt=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t}e^{-j2\pi ft}dt=\begin{cases}-j & (f>0)\\ 0 & (f=0)\\ j & (f<0)\end{cases}$$

# Representation of Band-Pass Signals and Systems

- We observe that $|H(f)|=1$ and the phase response $\Theta(f)=-\pi/2$ for $f>0$ and $\Theta(f)=\pi/2$ for $f<0$. Thus, this filter is basically a 90 degrees phase shifter for all frequencies in the input signal.
- The analytic signal $s_+(t)$ is a band-pass signal. To obtain an equivalent low-pass representation, we define:

$$S_l(f) = S_+(f + f_c)$$

$$s_l(t) = s_+(t)e^{-j2\pi f_c t} = \left[s(t) + j\hat{s}(t)\right]e^{-j2\pi f_c t}$$

- In general, $s_l(t)$ is complex-valued:

$$s_l(t) = x(t) + jy(t)$$

$$s(t) = x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t$$

$$\hat{s}(t) = x(t)\sin 2\pi f_c t + y(t)\cos 2\pi f_c t$$

# Representation of Band-Pass Signals and Systems

- $s(t)=x(t)\cos 2\pi f_c t-y(t)\sin 2\pi f_c t$ is the desired form for the representation of a band-pass signal. The low-frequency signal components $x(t)$ and $y(t)$ may be viewed as amplitude modulations impressed on the carrier components $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$, respectively.
- $x(t)$ and $y(t)$ are called the *quadrature components* of the band-pass signal $s(t)$.
- $s(t)$ can also be written as:

$$\left[s(t)+j\hat{s}(t)\right]=s_l(t)e^{j2\pi f_c t}$$

$$s(t)=\mathrm{Re}\left\{\left[x(t)+jy(t)\right]e^{j2\pi f_c t}\right\}=\mathrm{Re}\left[s_l(t)e^{j2\pi f_c t}\right]$$

- The low pass signal $s_l(t)$ is usually called the *complex envelope* of the real signal $s(t)$ and is basically the *equivalent low-pass signal*.

# Representation of Band-Pass Signals and Systems

- $s_l(t)$ can be also be written as:

$$s_l(t) = a(t)e^{j\theta(t)}$$

where $a(t) = \sqrt{x^2(t) + y^2(t)}$ and $\theta(t) = \tan^{-1}\dfrac{y(t)}{x(t)}$

- $s(t)$ can be represented as:

$$\begin{aligned} s(t) &= \mathrm{Re}\left[s_l(t)e^{j2\pi f_c t}\right] = \mathrm{Re}\left[a(t)e^{j[2\pi f_c t + \theta(t)]}\right] \\ &= a(t)\cos\left[2\pi f_c t + \theta(t)\right] \end{aligned}$$

$a(t)$ is called the *envelope* of $s(t)$, and $\theta(t)$ is called the *phase* of $s(t)$.

# Representation of Band-Pass Signals and Systems

- Three equivalent representations of band-pass signals:

$$\begin{aligned} s(t) &= x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t \\ &= \mathrm{Re}\left[s_l(t)e^{j2\pi f_c t}\right] \\ &= a(t)\cos\left[2\pi f_c t + \theta(t)\right] \end{aligned}$$

- The Fourier transform of *s*(*t*) is:

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty}\left\{\mathrm{Re}\left[s_l(t)e^{j2\pi f_c t}\right]\right\}e^{-j2\pi ft}dt$$

$$\begin{aligned} S(f) &= \frac{1}{2}\int_{-\infty}^{\infty}\left[s_l(t)e^{j2\pi f_c t} + s_l^*(t)e^{-j2\pi f_c t}\right]e^{-j2\pi ft}dt \\ &= \frac{1}{2}\left[S_l(f-f_c) + S_l^*(-f-f_c)\right] \end{aligned}$$

$$\mathrm{Re}(\xi) = \frac{1}{2}\left(\xi + \xi^*\right)$$

# Representation of Band-Pass Signals and Systems

- The energy in the signal *s*(*t*) is defined as:

$$\varepsilon = \int_{-\infty}^{\infty} s^2(t)\,dt = \int_{-\infty}^{\infty} \left\{ \mathrm{Re}\left[ s_l(t) e^{j2\pi f_c t} \right] \right\}^2 dt$$

$$\boxed{\mathrm{Re}(\xi) = \frac{1}{2}(\xi + \xi^*)}$$

$$\varepsilon = \frac{1}{4}\int_{-\infty}^{\infty} \left[ s_l^2 e^{j4\pi f_c t} + 2 s_l s_l^* + \left(s_l^*\right)^2 e^{-j4\pi f_c t} \right] dt$$

$$\boxed{s_l(t) = a(t) e^{j\theta(t)}}$$

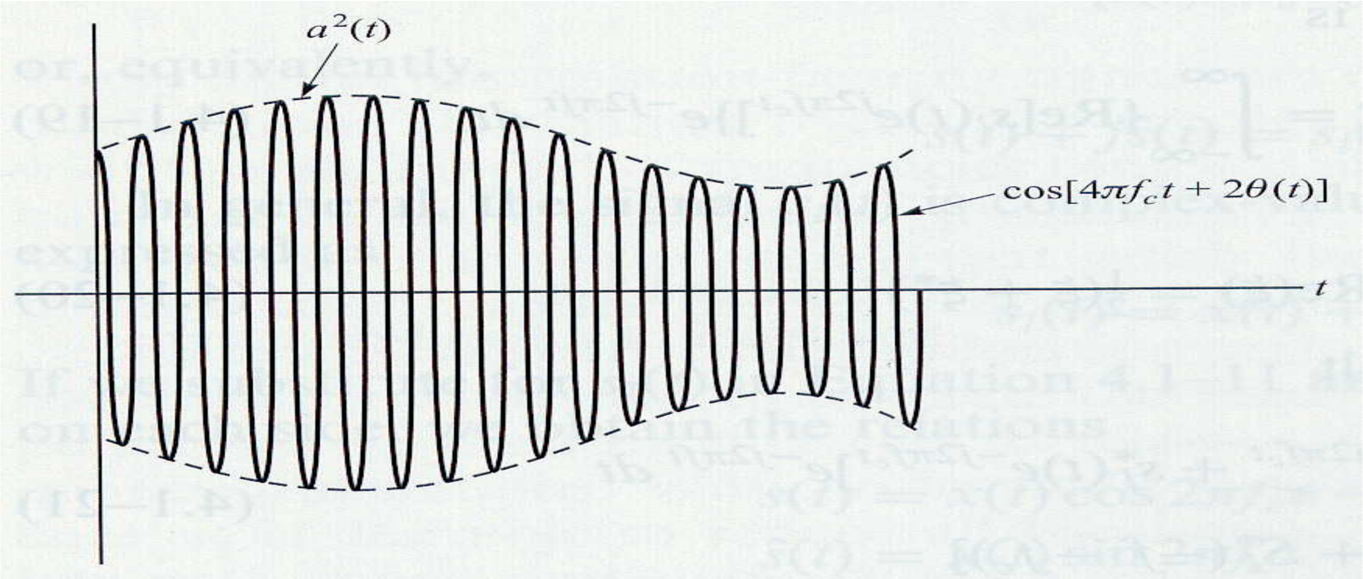
$$= \frac{1}{2}\int_{-\infty}^{\infty} \left| s_l(t) \right|^2 dt + \frac{1}{4}\int_{-\infty}^{\infty} \left[ a^2(t) e^{j4\pi f_c t + 2\theta(t)} + \left( a^*(t) \right)^2 e^{-(j4\pi f_c t + 2\theta(t))} \right] dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left| s_l(t) \right|^2 dt + \frac{1}{2}\int_{-\infty}^{\infty} \left| s_l(t) \right|^2 \cos\left[ 4\pi f_c t + 2\theta(t) \right] dt$$

where $\left| s_l(t) \right|^2 = a^2(t) = \left( a^*(t) \right)^2$

# Representation of Band-Pass Signals and Systems

- Since the signal $s(t)$ is narrow-band, the real envelope $a(t)=|s_l(t)|$ or, equivalently, $a^2(t)$ varies slowly relative to the rapid variations exhibited by the cosine function.



- The net area contributed by the second integral is very small relative to the value of the first integral, hence, it can be neglected.

$$\varepsilon = \frac{1}{2}\int_{-\infty}^{\infty} \left| s_l\left(t\right) \right|^2 dt$$

# Representation of Linear Band-Pass Systems

- A linear filter or system may be described either by its impulse response $h(t)$ or by its frequency response $H(f)$, which is the Fourier transform of $h(t)$. Since $h(t)$ is real, $H^*(-f)=H(f)$, because:

$$H^*(-f)=\left(\int_{-\infty}^{\infty} h(t)e^{-j2\pi(-f)t}dt\right)^*$$

$$=\int_{-\infty}^{\infty} h^*(t)e^{-j2\pi ft}dt \qquad \text{Note that } h(t) \text{ is real.}$$

$$=\int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt=H(f)$$

- Define $H_l(f-f_c)=\begin{cases} H(f) & (f>0) \\ 0 & (f<0) \end{cases}$

then $H_l^*(-f-f_c)=\begin{cases} 0 & (f>0) \\ H^*(-f)=H(f) & (f<0) \end{cases}$

# Representation of Linear Band-Pass Systems

- As a result $H(f)=H_l(f-f_c)+H_l^*(-f-f_c)$

  thus
  $$\begin{aligned} h(t) &= h_l(t)e^{j2\pi f_c t}+h_l^* e^{-j2\pi f_c t} \\ &= 2\operatorname{Re}\left[h_l(t)e^{j2\pi f_c t}\right] \end{aligned}$$

  where $\int_{-\infty}^{\infty} H_l^*(-f-f_c)e^{j2\pi ft}df$ using $x=-f-f_c$

  $$=\int_{-\infty}^{\infty} H_l^*(x)e^{-j2\pi xt}dx\cdot e^{-j2\pi f_c t}$$

  $$=\left(\int_{-\infty}^{\infty} H_l(x)e^{j2\pi xt}dx\right)^*\cdot e^{-j2\pi f_c t}=h_l^*(t)\cdot e^{-j2\pi f_c t}$$

- $h_l(t)$ is the inverse Fourier transform of $H_l(f)$.
- In general, the impulse response $h_l(t)$ of the equivalent low-pass system is complex-valued.

# Representation of Linear Band-Pass Systems

- We have shown that narrowband band-pass signals and systems can be represented by equivalent low-pass signals and systems.
- We demonstrate in this section that the output of a band-pass system to a band-pass input signal is simply obtained from the equivalent low-pass input signal and the equivalent low-pass impulse response of the system.
- The output of the band-pass system is also a band-pass signal, and, therefore, it can be expressed in the form:

$$r(t) = \mathrm{Re}\left[ r_l(t) e^{j2\pi f_c t} \right]$$

where $r(t)$ is related to the input signal $s(t)$ and the impulse response $h(t)$ by the convolution integral.

$$r(t) = \int_{-\infty}^{\infty} s(\tau) h(t-\tau) d\tau$$

# Response of a Band-Pass System to a Band-Pass Signal

- The output of the system in the frequency domain is:

$$R(f)=S(f)H(f)$$

P.10

P.14

$$=\frac{1}{2}\left[S_l(f-f_c)+S_l^*(-f-f_c)\right]\left[H_l(f-f_c)+H_l^*(-f-f_c)\right]$$

  - For a narrow band signal, $S_l(f-f_c)\approx 0$ for $f<0$ and $H^*_l(-f-f_c)=0$ for $f>0$.

$$S_l(f-f_c)H_l^*(-f-f_c)=0$$

P.13

  - For a narrow band signal, $S^*_l(-f-f_c)\approx 0$ for $f>0$ and $H_l(f-f_c)=0$ for $f<0$.

$$S_l^*(-f-f_c)H_l(f-f_c)=0$$

P.13

$$R(f)=\frac{1}{2}\left[S_l(f-f_c)H_l(f-f_c)+S_l^*(-f-f_c)H_l^*(-f-f_c)\right]$$

$$=\boxed{\frac{1}{2}\left[R_l(f-f_c)+R_l^*(-f-f_c)\right]}$$

$$R_l(f)\equiv S_l(f)H_l(f)$$

$$r_l(t)=\int_{-\infty}^{\infty}s_l(\tau)h_l(t-\tau)d\tau$$

# Representation of Band-Pass Stationary Stochastic Processes

- In this section, we extend the representation to sample functions of a *band-pass stationary stochastic process*. In particular, we derive the relations between the correlation functions and power spectra of the band-pass signal and the correlation function and power spectra of the equivalent low-pass signal.
- Suppose that $n(t)$ is a sample function of a wide-sense stationary stochastic process with zero mean and power spectral density $\Phi_{nn}(f)$. The power spectral density is assumed to be zero outside of an interval of frequencies centered around $f_c$, where $f_c$ is termed the *carrier frequency*. The stochastic process $n(t)$ is said to be a *narrowband band-pass process* if the width of the spectral density is much smaller than $f_c$.

# Representation of Band-Pass Stationary Stochastic Processes

- Under this condition, a sample function of the process $n(t)$ can be represented by the following equations:

$$\boxed{\begin{aligned} n(t) &= a(t)\cos\left[2\pi f_c t + \theta(t)\right] \\ &= x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t \\ &= \operatorname{Re}\left[z(t)e^{j2\pi f_c t}\right] \end{aligned}}$$

  - $a(t)$ is the envelope and $\theta(t)$ is the phase of the real-valued signal.
  - $x(t)$ and $y(t)$ are the quadrature components of $n(t)$.
  - $z(t)$ is called the complex envelope of $n(t)$.
  - If $n(t)$ is zero mean, then $x(t)$ and $y(t)$ must also have zero mean values.
  - The stationarity of $n(t)$ implies that:

$$\phi_{xx}(\tau) = \phi_{yy}(\tau)$$

Proved next.

$$\phi_{xy}(\tau) = -\phi_{yx}(\tau)$$

# Representation of Band-Pass Stationary Stochastic Processes

- Proof of $\phi_{xx}(\tau)=\phi_{yy}(\tau)$ and $\phi_{xy}(\tau)=-\phi_{yx}(\tau)$

  Autocorrelation function of $n(t)$ is:

$$\phi_{nn}(\tau)=E\left[n(t)n(t+\tau)\right]$$

$$=E\left\{\left[x(t)\cos 2\pi f_c t-y(t)\sin 2\pi f_c t\right]\right.$$

$$\left.\times\left[x(t+\tau)\cos 2\pi f_c(t+\tau)-y(t+\tau)\sin 2\pi f_c(t+\tau)\right]\right\}$$

$$=\phi_{xx}(\tau)\cos 2\pi f_c t\cos 2\pi f_c(t+\tau)+\phi_{yy}(\tau)\sin 2\pi f_c t\sin 2\pi f_c(t+\tau)$$

$$-\phi_{xy}(\tau)\sin 2\pi f_c t\cos 2\pi f_c(t+\tau)-\phi_{yx}(\tau)\cos 2\pi f_c t\sin 2\pi f_c(t+\tau)$$

by using:

$$\cos A\cos B=\tfrac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

$$\sin A\sin B=\tfrac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$

$$\sin A\cos B=\tfrac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$$

# Representation of Band-Pass Stationary Stochastic Processes

We can obtain:

$$\phi_{nn}(\tau) = E\left[n(t)n(t+\tau)\right]$$

$$= \frac{1}{2}\left[\phi_{xx}(\tau)+\phi_{yy}(\tau)\right]\cos 2\pi f_c\tau + \frac{1}{2}\left[\phi_{xx}(\tau)-\phi_{yy}(\tau)\right]\cos 2\pi f_c(2t+\tau)$$

$$-\frac{1}{2}\left[\phi_{yx}(\tau)-\phi_{xy}(\tau)\right]\sin 2\pi f_c\tau - \frac{1}{2}\left[\phi_{yx}(\tau)+\phi_{xy}(\tau)\right]\sin 2\pi f_c(2t+\tau)$$

Since *n*(*t*) is stationary, the right-hand side must be independent of *t*.

As a result, $\phi_{xx}(\tau) = \phi_{yy}(\tau)$ and $\phi_{xy}(\tau) = -\phi_{yx}(\tau)$ Q.E.D.

Therefore,

$$\phi_{nn}(\tau) = \phi_{xx}(\tau)\cos 2\pi f_c\tau - \phi_{yx}(\tau)\sin 2\pi f_c\tau$$

Note that this equation is identical in form to:

$$n(t) = x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t$$

# Representation of Band-Pass Stationary Stochastic Processes

- The autocorrelation function of the equivalent low-pass process $z(t)=x(t)+jy(t)$ is defined as:

$$\phi_{zz}(\tau)=\frac{1}{2}E\left[z^*(t)z(t+\tau)\right]$$

$$\phi_{zz}(\tau)=\frac{1}{2}\left[\phi_{xx}(\tau)+\phi_{yy}(\tau)-j\phi_{xy}(\tau)+j\phi_{yx}(\tau)\right]$$

Since $\varphi_{xx}(\tau)=\varphi_{yy}(\tau)$ and $\varphi_{xy}(\tau)= -\varphi_{yx}(\tau)$

we obtain: $\boxed{\varphi_{zz}(\tau)=\varphi_{xx}(\tau)+ j\varphi_{yx}(\tau)}$

- This equation relates the autocorrelation function of the complex envelope to the autocorrelation and cross-correlation functions of the quadrature components.

# Representation of Band-Pass Stationary Stochastic Processes

- By combining $\phi_{zz}(\tau) = \phi_{xx}(\tau) + j\phi_{yx}(\tau)$

  and $\phi_{nn}(\tau) = \phi_{xx}(\tau)\cos 2\pi f_c\tau - \phi_{yx}(\tau)\sin 2\pi f_c\tau$

  we can obtain: $\boxed{\varphi_{nn}(\tau)=\text{Re}[\varphi_{zz}(\tau)e^{j2\pi fc\tau}]}$

- Therefore, the autocorrelation function $\varphi_{nn}(\tau)$ of the band-pass stochastic process is uniquely determined from the autocorrelation function $\varphi_{zz}(\tau)$ of the equivalent low-pass process $z(t)$ and the carrier frequency $f_c$.

- The power density spectrum of the stochastic process $n(t)$ is:

$$\boxed{\begin{aligned}\Phi_{nn}(f) &= \int_{-\infty}^{\infty}\left\{\text{Re}\left[\phi_{zz}(\tau)e^{j2\pi f_c\tau}\right]\right\}e^{-j2\pi f\tau}d\tau \\ &= \frac{1}{2}\left[\Phi_{zz}(f-f_c)+\Phi_{zz}(-f-f_c)\right]\end{aligned}}$$

# Representation of Band-Pass Stationary Stochastic Processes

- Properties of the quadrature components
  - $\phi_{yx}(\tau) = \phi_{xy}(-\tau)$ (Ch2, P.99)

    $\phi_{xy}(\tau) = -\phi_{yx}(\tau)$ (P.18)

    $\Rightarrow \quad \phi_{xy}(\tau) = -\phi_{xy}(-\tau)$

    $\Rightarrow \quad \phi_{xy}(\tau)$ is an odd function of $\tau$ => $\phi_{xy}(0) = 0$

    =>$x$,$y$ uncorrelated for $\tau$=0

  - If $\phi_{xy}(\tau) = 0$ for all $\tau$, then $\phi_{zz}(\tau)$ is real (from page 21) and the power spectral density satisfies $\Phi_{zz}(f) = \Phi_{zz}(-f)$ (i.e. $\Phi_{zz}(f)$ is symmetric about $f = 0$).

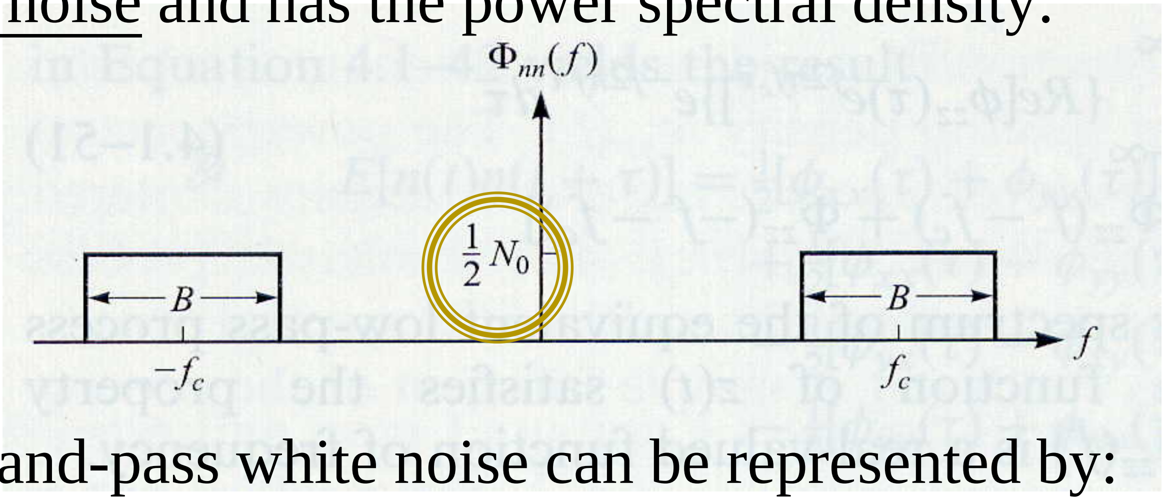
$\varphi_{zz}(\tau)=\varphi_{xx}(\tau)+ j\varphi_{yx}(\tau)$

# Representation of Band-Pass Stationary Stochastic Processes

- Representation of white noise
  - White noise is a stochastic process that is defined to have a flat (constant) power spectral density over the entire frequency range. This type of noise can’t be expressed in terms of quadrature components, as a result of its wideband character.
  - In the demodulation of narrowband signals in noise, it is mathematically convenient to model the additive noise process as white and to represent the noise in terms of quadrature components. This can be accomplished by postulating that the signals and noise at the receiving terminal have passed through an ideal band-pass filter.

# Representation of Band-Pass Stationary Stochastic Processes

- Representation of white noise (cont.)
  - The noise resulting from passing the white noise process through a spectrally band-pass filter is termed band-pass white noise and has the power spectral density:



  - The band-pass white noise can be represented by:

$$\begin{aligned} n(t) &= a(t)\cos\left[2\pi f_c t + \theta(t)\right] \\ &= x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t \\ &= \operatorname{Re}\left[z(t)e^{j2\pi f_c t}\right] \end{aligned}$$

# Representation of Band-Pass Stationary Stochastic Processes

- Representation of white noise (cont.)
  - The equivalent low-pass noise $z(t)$ has a power spectral density:

$$\Phi_{zz}(f)=\begin{cases} N_0 & \left(|f|\le\frac{1}{2}B\right) \\ 0 & \left(|f|>\frac{1}{2}B\right)\end{cases}$$

P.23

$$\phi_{zz}(\tau)=N_0\frac{\sin\pi B\tau}{\pi\tau} \quad \text{and} \quad \phi_{zz}(\tau)\underset{B\to\infty}{=}N_0\delta(\tau)$$

  - The power spectral density for white noise and band-pass white noise is symmetric about $f=0$, so $\varphi_{yx}(\tau)=0$ for all $\tau$.

$$\phi_{zz}(\tau)=\phi_{xx}(\tau)=\phi_{yy}(\tau)$$

$$\boxed{\left(\because \phi_{zz}(\tau)=\frac{1}{2}\left[\phi_{xx}(\tau)+\phi_{yy}(\tau)-j\phi_{xy}(\tau)+j\phi_{yx}(\tau)\right]\right)}$$

# Vector Space Concepts

- We will demonstrate that signals have characteristics that are similar to vectors and develop a vector representation for signal waveforms.
- Vector Space Concepts
  - A vector **v** in an *n*-dimensional space is characterized by its *n* components $[v_1\ v_2\ \ldots\ v_n]$ and may also be represented as a linear combination of *unit vectors* or *basis vectors* $\mathbf{e}_i$, $1 \le i \le n$,

$$\mathbf{v} = \sum_{i=1}^{n} v_i e_i$$

  - The *inner product* of two *n*-dimensional vectors is defined as:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum_{i=1}^{n} v_{1i} v_{2i}$$

# Vector Space Concepts

- A set of *m* vectors $\mathbf{v}_k$, $1\le k\le m$ are orthogonal if:

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \text{ for all } 1 \le i,\ j \le m, \text{ and } i \ne j.$$

- The *norm* of a vector **v** is denoted by $\|\mathbf{v}\|$ and is defined as:

$$\|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

- A set of *m* vectors is said to be *orthonormal* if the vectors are orthogonal and each vector has a unit norm.
- A set of *m* vectors is said to be *linearly independent* if no one vector can be represented as a linear combination of the remaining vectors.
- Two *n*-dimensional vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy the *triangle inequality*:

$$\|\mathbf{v}_1 + \mathbf{v}_2\| \le \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$$

# Vector Space Concepts

- *Cauchy-Schwarz inequality*:

$$\left|\mathbf{v}_1 \cdot \mathbf{v}_2\right| \leq \left\|\mathbf{v}_1\right\| \cdot \left\|\mathbf{v}_2\right\|$$

- The norm square of the sum of two vectors may be expressed as:

$$\left\|\mathbf{v}_1 + \mathbf{v}_2\right\|^2 = \left\|\mathbf{v}_1\right\|^2 + \left\|\mathbf{v}_2\right\|^2 + 2\mathbf{v}_1 \cdot \mathbf{v}_2$$

- *Linear transformation* in an *n*-dimensional vector space:

$$\mathbf{v}' = A\mathbf{v}$$

- In the special case where $\mathbf{v}'=\lambda\mathbf{v}$, $A\mathbf{v} = \lambda\mathbf{v}$

  the vector **v** is called an *eigenvector* and λ is the corresponding *eigenvalue*.

# Vector Space Concepts

- *Gram-Schmidt procedure* for constructing a set of orthonormal vectors.
  - Arbitrarily selecting a vector $\mathbf{v}_1$ and normalizing its length:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\left\|\mathbf{v}_1\right\|}$$

  - Select $\mathbf{v}_2$ and subtract the projection of $\mathbf{v}_2$ onto $\mathbf{u}_1$.

$$\mathbf{u}_2^{'} = \mathbf{v}_2 - \left(\mathbf{v}_2 \cdot \mathbf{u}_1\right)\mathbf{u}_1$$

  - Normalize the vector $\mathbf{u}_2$’ to unit length. $\mathbf{u}_2 = \frac{\mathbf{u}_2^{'}}{\left\|\mathbf{u}_2^{'}\right\|}$
  - Selecting $\mathbf{v}_3$: $\mathbf{u}_3^{'} = \mathbf{v}_3 - \left(\mathbf{v}_3 \cdot \mathbf{u}_1\right)\mathbf{u}_1 - \left(\mathbf{v}_3 \cdot \mathbf{u}_2\right)\mathbf{u}_2$ $\quad \mathbf{u}_3 = \frac{\mathbf{u}_3^{'}}{\left\|\mathbf{u}_3^{'}\right\|}$
  - By continuing this procedure, we construct a set of orthonormal vectors.

# Signal Space Concepts

- The *inner product* of two generally complex-valued signals $x_1(t)$ and $x_2(t)$ is denote by < $x_1(t)$,$x_2(t)$> and defined as:

$$\left\langle x_1(t)\ ,x_2(t)\right\rangle = \int_a^b x_1(t)\,x_2^*(t)\,dt$$

- The signals are *orthogonal* if their inner product is zero.
- The *norm* of a signal is defined as:

$$\left\| x(t) \right\| = \left( \int_a^b \left| x(t) \right|^2 dt \right)^{1/2}$$

- A set of *m* signals are *orthonormal* if they are orthogonal and their norms are all unity.
- A set of *m* signals is *linearly independent*, if no signal can be represented as a linear combination of the remaining signals.

# Signal Space Concepts

- The *triangle inequality* for two signals is:

$$\left\| x_1(t) + x_2(t) \right\| \le \left\| x_1(t) \right\| + \left\| x_2(t) \right\|$$

- The *Cauchy-Schwarz inequality* is:

$$\left| \int_a^b x_1(t)\, x_2^*(t)\, dt \right| \le \left| \int_a^b \left| x_1(t) \right|^2 dt \right|^{1/2} \left| \int_a^b \left| x_2(t) \right|^2 dt \right|^{1/2}$$

with equality when $x_2(t)=ax_1(t)$, where $a$ is any complex number.

# Orthogonal Expansions of Signals

- Suppose that $s(t)$ is a deterministic, real-valued signal with finite energy:

$$\varepsilon_s = \int_{-\infty}^{\infty} \left[ s(t) \right]^2 dt$$

- Suppose that there exists a set of functions $\{f_n(t), n=1,2,\ldots,K\}$ that are *orthonormal* in the sense that:

$$\int_{-\infty}^{\infty} f_n(t) f_m(t) dt = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases}$$

- We may approximate the signal $s(t)$ by a weighted linear combination of these functions, i.e.,

$$\overset{\wedge}{s}(t) = \sum_{k=1}^{K} s_k f_k(t)$$

# Orthogonal Expansions of Signals

- The approximation error incurred is:

$$e(t) = s(t) - \hat{s}(t)$$

- The energy of the approximation error:

$$\varepsilon_e = \int_{-\infty}^{\infty} \left[ s(t) - \hat{s}(t) \right]^2 dt = \int_{-\infty}^{\infty} \left[ s(t) - \sum_{k=1}^{K} s_k f_k(t) \right]^2 dt \qquad (*)$$

- To minimize the energy of the approximation error, the optimum coefficients in the series expansion of $s(t)$ may be found by:
  - Differentiating Equation (*) with respect to each of the coefficients $\{s_k\}$ and setting the first derivatives to zero.
  - Use a well-known result from estimation theory based on the mean-square-error criterion, which is that the minimum of $\varepsilon_e$ with respect to the $\{s_k\}$ is obtained when the error is orthogonal to each of the functions in the series expansion.

# Orthogonal Expansions of Signals

- Using the second approach, we have:

$$\int_{-\infty}^{\infty}\left[s(t)-\sum_{k-1}^{K} s_k f_k(t)\right] f_n(t)\,dt = 0, \quad n = 1, 2, \ldots, K$$

- Since the functions $\{f_n(t)\}$ are orthonormal, we have:

$$s_n = \int_{-\infty}^{\infty} s(t)\, f_n(t)\,dt, \quad n = 1, 2, \ldots, K$$

  Thus, the coefficients are obtained by projecting the signals $s(t)$ onto each of the functions.

- The minimum mean square approximation error is:

$$\varepsilon_{\min} = \int_{-\infty}^{\infty} e(t)\,s(t)\,dt = \int_{-\infty}^{\infty}\left[s(t)\right]^2 dt - \int_{-\infty}^{\infty}\sum_{k=1}^{K} s_k f_k(t)\,s(t)\,dt$$

$$= \varepsilon_s - \sum_{k=1}^{K} s_k^2$$

# Orthogonal Expansions of Signals

- When the minimum mean square approximation error $\varepsilon_{\min}=0$,

$$\varepsilon_s = \sum_{k=1}^{K} s_k^2 = \int_{-\infty}^{\infty} \left[ s(t) \right]^2 dt$$

- Under such condition, we may express $s(t)$ as:

$$s(t) = \sum_{k=1}^{K} s_k f_k(t)$$

- When every finite energy signal can be represented by a series expansion of the form for which $\varepsilon_{\min}=0$, the set of orthonormal functions $\{f_n(t)\}$ is said to be *complete*.

# Orthogonal Expansions of Signals

- Gram-Schmidt procedure
  - Constructing a set of orthonormal waveforms from a set of finite energy signal waveforms $\{s_i(t), i=1,2,\ldots,M\}$.
  - Begin with the first waveform $s_1(t)$ which has energy $\varepsilon_1$. The first orthonormal waveform is:

$$f_1(t) = \frac{s_1(t)}{\sqrt{\varepsilon_1}}$$

  - The 2*nd* waveform is constructed from $s_2(t)$ by first computing the projection of $f_1(t)$ onto $s_2(t)$:

$$c_{12} = \int_{-\infty}^{\infty} s_2(t) f_1(t)\, dt$$

  - Then $c_{12} f_1(t)$ is subtracted from $s_2(t)$:

$$f_2^{'}(t) = s_2(t) - c_{12} f_1(t)$$

# Orthogonal Expansions of Signals

- Gram-Schmidt procedure (cont.)
  - If ε$_2$ denotes the energy of $f_2'(t)$, the normalized waveform that is orthogonal to $f_1(t)$ is:

$$f_2(t) = \frac{f_2^{'}(t)}{\sqrt{\varepsilon_2}}$$

  - In general, the orthogonalization of the *k*th function leads to

$$f_k(t) = \frac{f_k^{'}(t)}{\sqrt{\varepsilon_k}} \quad \text{where} \quad f_k^{'}(t) = s_k(T) - \sum_{i=1}^{k-1} c_{ik} f_i(t)$$

$$c_{ik} = \int_{-\infty}^{\infty} s_k(t) f_i(t)\,dt, \quad i = 1, 2, \ldots, k-1$$

  - The orthogonalization process is continued until all the *M* signal waveforms have been exhausted and $N \leq M$ orthonormal waveforms have been constructed.

# Orthogonal Expansions of Signals

- Gram-Schmidt procedure (cont.)
  - Once we have constructed the set of orthonormal waveforms $\{f_n(t)\}$, we can express the $M$ signals $\{s_n(t)\}$ as linear combinations of the $\{f_n(t)\}$:

$$s_k(t) = \sum_{n=1}^{N} s_{kn} f_n(t), \quad k = 1, 2, \ldots, M$$

$$\varepsilon_k = \int_{-\infty}^{\infty} \left[ s_k(t) \right]^2 dt = \sum_{n=1}^{N} s_{kn}^2 = \left\| s_k \right\|^2$$

$$s_k = \left[ s_{k1} \ s_{k2} \ \ldots \ s_{kN} \right]$$

  - Each signal may be represented as a point in the $N$-dimensional signal space with coordinates $\{s_{ki}, i=1,2,\ldots,N\}$.

# Orthogonal Expansions of Signals

- Gram-Schmidt procedure (cont.)
  - The energy in the *k*th signal is simply the square of the length of the vector or, equivalently, the square of the Euclidean distance from the origin to the point in the $N$-dimensional space.
  - Any signal can be represented geometrically as a point in the signal space spanned by the $\{f_n(t)\}$.
  - The functions $\{f_n(t)\}$ obtained from the Gram-Schmidt procedure are not unique.
  - If we alter the order in which the orthogonalization of the signals $\{s_n(t)\}$ is performed, the orthonormal waveforms will be different.
  - Nevertheless, the vectors $\{s_n(t)\}$ will retain their geometrical configuration and their lengths will be invariant to the choice of orthonormal functions $\{f_n(t)\}$.

# Orthogonal Expansions of Signals

- Consider the case in which the signal waveforms are band-pass and represented as:

$$s_m(t) = \mathrm{Re}\left[ s_{lm}(t) e^{j2\pi f_c t} \right], \qquad m = 1, 2, \ldots, M$$

$$\varepsilon_m = \int_{-\infty}^{\infty} s_m^2(t)\, dt = \frac{1}{2} \int_{-\infty}^{\infty} \left| s_{lm}(t) \right|^2 dt \qquad \text{(Page 12)}$$

- Similarity between any pair of signal waveforms is measured by the *normalized cross correlation*:

$$\frac{1}{\sqrt{\varepsilon_m \varepsilon_k}} \int_{-\infty}^{\infty} s_m(t) s_k(t)\, dt = \mathrm{Re}\left\{ \frac{1}{2\sqrt{\varepsilon_m \varepsilon_k}} \int_{-\infty}^{\infty} s_{lm}(t) s_{lk}^*(t)\, dt \right\}$$

- *Complex-valued cross-correlation coefficient* $\rho_{km}$ is defined as:

$$\rho_{km} = \frac{1}{2\sqrt{\varepsilon_m \varepsilon_k}} \int_{-\infty}^{\infty} s_{lm}^*(t) s_{lk}(t)\, dt$$

# Orthogonal Expansions of Signals

- $$\operatorname{Re}(\rho_{km}) = \frac{1}{\sqrt{\varepsilon_m \varepsilon_k}} \int_{-\infty}^{\infty} s_m(t) s_k(t)\, dt$$

$$\operatorname{Re}(\rho_{km}) = \frac{s_m \cdot s_k}{\|s_m\| \|s_k\|} = \frac{s_m \cdot s_k}{\sqrt{\varepsilon_m \varepsilon_k}}$$

- The *Euclidean distance* between a pair of signals is defined as:

$$d_{km}^{(e)} = \|s_m - s_k\| = \left\{ \int_{-\infty}^{\infty} \left[ s_m(t) - s_k(t) \right]^2 dt \right\}^{1/2}$$

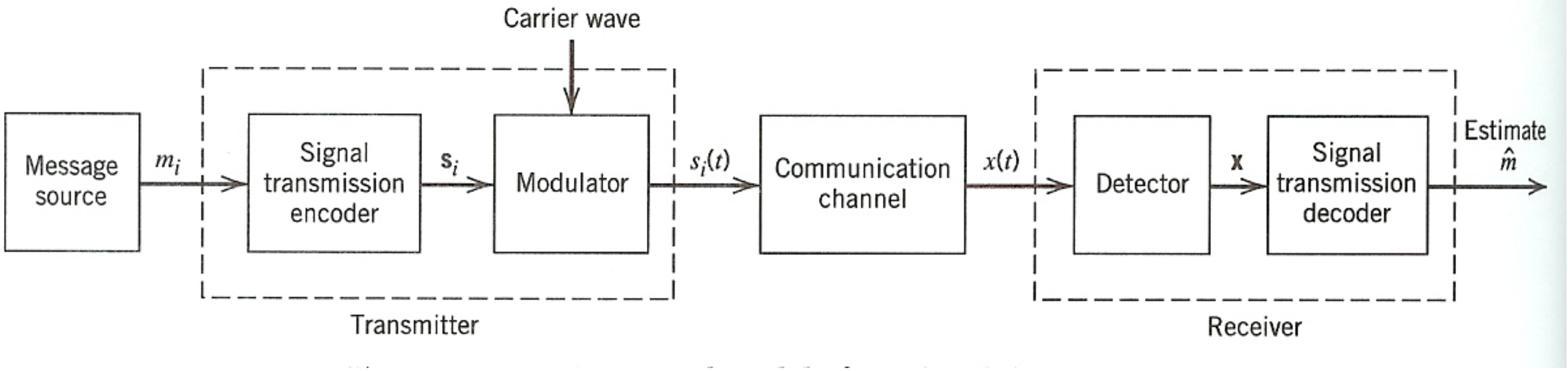
$$= \left\{ \varepsilon_m + \varepsilon_k - 2\sqrt{\varepsilon_m \varepsilon_k} \operatorname{Re}(\rho_{km}) \right\}^{1/2}$$

- When $\varepsilon_m = \varepsilon_k = \varepsilon$ for all *m* and *k*, this expression simplifies to:

$$d_{km}^{(e)} = \left\{ 2\varepsilon \left[ 1 - \operatorname{Re}(\rho_{km}) \right] \right\}^{1/2}$$

# Representation of Digitally Modulated Signals

- In the transmission of digital information over a communication channel, the *modulator* is the interface device that maps the digital information into analog waveforms that match the characteristics of the channel.
- The mapping is generally performed by taking blocks of $k=\log_2 M$ binary digits at a time from the information sequence $\{a_n\}$ and selecting one of $M=2^k$ deterministic, finite energy waveforms $\{s_m(t), m=1,2,\cdots,M\}$ for transmission over the channel.
- Functional model of passband data transmission system

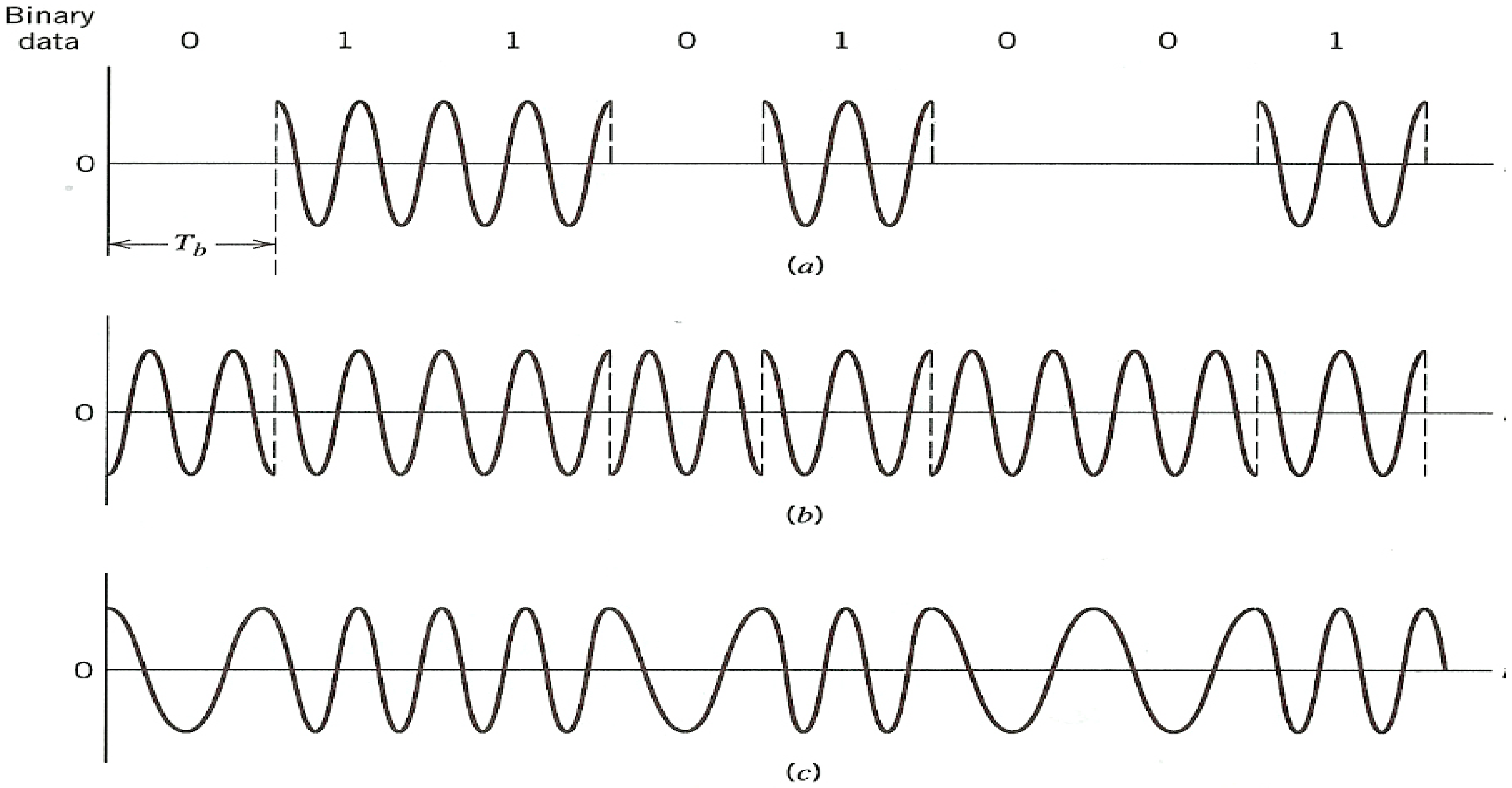


# Representation of Digitally Modulated Signals

- The digital data transmits over a band-pass channel that can be *linear* or *nonlinear*.
- When the mapping is performed under the constraint that a waveform transmitted in any time interval depends on one or more previously transmitted waveforms, the modulator is said to have *memory*. Otherwise, the modulator is called *memoryless*.
- In digital passband transmission, the incoming data stream is modulated onto a carrier (usually sinusoidal) with fixed frequency limits imposed by a band-pass channel of interest.
- The modulation process making the transmission possible involves switching (keying) the amplitude, frequency, or phase of a sinusoidal carrier in some fashion in accordance with the incoming data.
- There are three basic signaling schemes: amplitude-shift keying (ASK), frequency-shift Keying (FSK), and phase-shift keying (PSK).

# Representation of Digitally Modulated Signals

- Illustrative waveforms for the three basic forms of signaling binary information. (*a*) ASK (*b*) PSK (*c*) FSK.



# Representation of Digitally Modulated Signals

- Unlike ASK signals, both PSK and FSK signals have a constant envelope. This property makes PSK and FSK signals impervious to amplitude nonlinearities.
- In practice, we find that PSK and FSK signals are preferred to ASK signals for passband data transmission over nonlinear channels.
- Digital modulation techniques may be classified into *coherent* and *noncoherent* techniques, depending on whether the receiver is equipped with a *phase-recovery circuit* or not.
- The *phase-recovery circuit* ensures that the oscillator supplying the locally generated carrier wave in the receiver is synchronized (in both frequency and phase) to the oscillator supplying the carrier wave used to originally modulated the incoming data stream in the transmitter.

# Memoryless Modulation Methods

- Pulse-amplitude-modulated (PAM) signals
  - *Double-sideband (DSB)* signal waveform may be represented as:

$$s_m(t) = \operatorname{Re}\left[A_m g(t) e^{j2\pi f_c t}\right]$$

Bandpass Signal

$$= A_m g(t) \cos 2\pi f_c t, \quad m = 1, 2, \ldots, M, \quad 0 \le t \le T$$

where $A_m$ denote the set of $M$ possible amplitudes corresponding to $M$=$2^k$ possible $k$-bit blocks of symbols.

  - The signal amplitudes $A_m$ take the discrete values:

$$A_m = (2m - 1 - M)d, \quad m = 1, 2, \ldots, M \quad \left(-(M-1)d \ldots (M-1)d\right)$$

  - $2d$ is the distance between adjacent signal amplitudes.
  - $g(t)$ is a real-valued signal pulse whose shape influences the spectrum of the transmitted signal.
  - The *symbol rate* is $R/k$, $T_b$=$1/R$ is the *bit interval*, and $T$=$k/R$=$kT_b$ is the *symbol interval*.

# Memoryless Modulation Methods

- Pulse-amplitude-modulated (PAM) signals (cont.)
  - The *M* PAM signals have energies:

$$\varepsilon_m = \int_0^T s_m^2(t)\,dt = \frac{1}{2}A_m^2 \int_0^T g^2(t)\,dt = \frac{1}{2}A_m^2 \varepsilon_g$$

  - These signals are one-dimensional and are represented by:

$$s_m(t) = s_m f(t)$$

  - *f*(*t*) is defined as the *unit-energy signal* waveform given as:

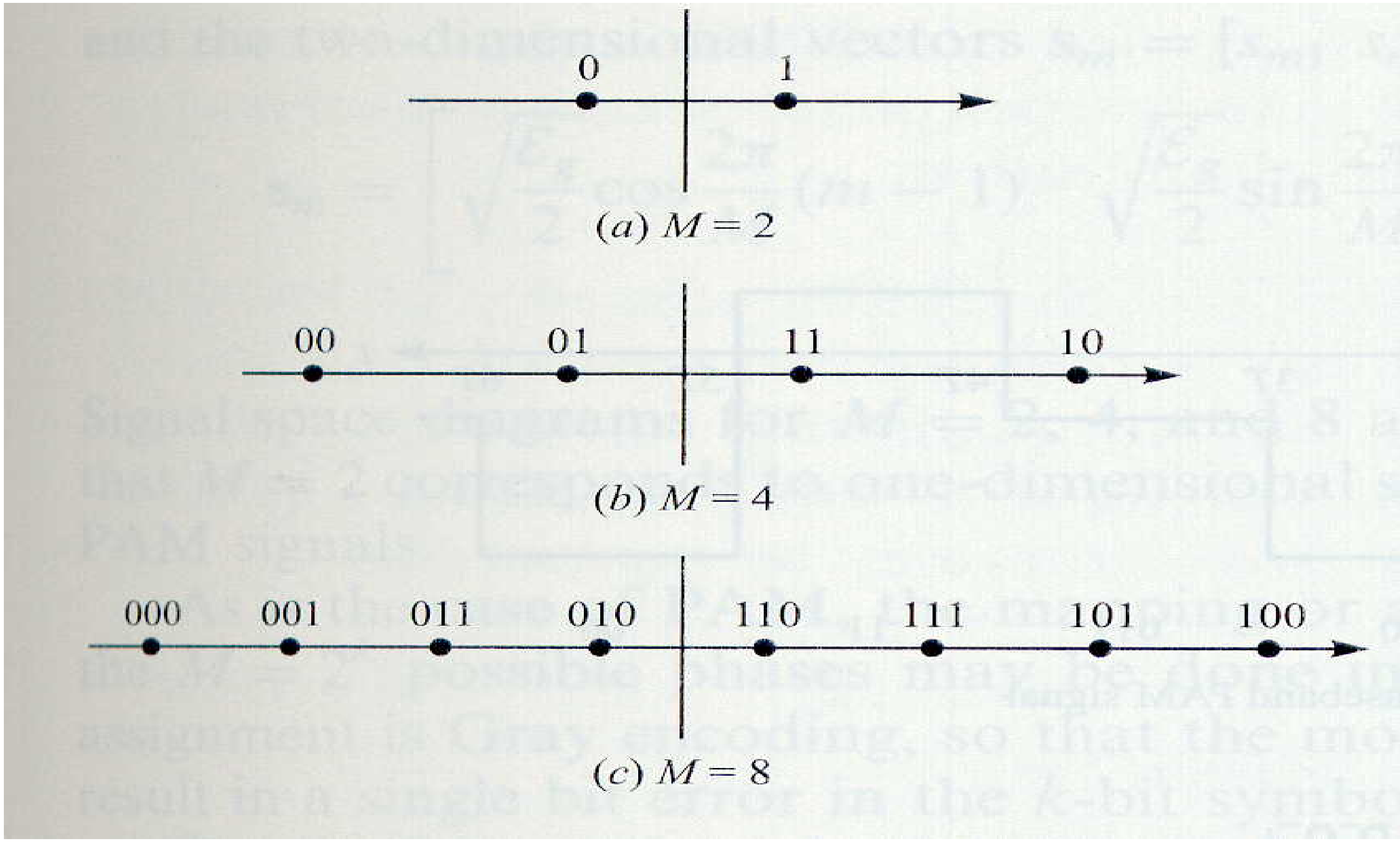
$$f(t) = \sqrt{\frac{2}{\varepsilon_g}}\, g(t) \cos 2\pi f_c t$$

$$s_m = A_m \sqrt{\frac{1}{2}\varepsilon_g}\,, \quad m = 1,2,\ldots,M$$

  - *Digital PAM* is also called *amplitude-shift keying (ASK)*.

# Memoryless Modulation Methods

- Pulse-amplitude-modulated (PAM) signals (cont.)
  - Signal space diagram for digital PAM signals:



# Memoryless Modulation Methods

- Pulse-amplitude-modulated (PAM) signals (cont.)
  - *Gray encoding*: The mapping of *k* information bits to the $M=2^k$ possible signal amplitudes may be done in a number of ways. The preferred assignment is one in which the adjacent signal amplitudes differ by one binary digit.
  - The *Euclidean distance* between any pair of signal points is:

$$d_{mn}^{(e)} = \sqrt{\left(s_m - s_n\right)^2} = \sqrt{\frac{1}{2}\varepsilon_g}\left|A_m - A_n\right| = d\sqrt{2\varepsilon_g}\left|m-n\right|$$

  - The *minimum Euclidean distance* between any pair of signals is:

$$d_{\min}^{(e)} = d\sqrt{2\varepsilon_g}$$

# Memoryless Modulation Methods

- Phase-modulated signals (Binary Phase-Shift Keying)
  - In a coherent binary PSK system, the pair of signals $s_1(t)$ and $s_2(t)$ used to represent binary symbols 1 and 0, respectively is defined by

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$$

    where $0 \le t \le T_b$ and $E_b$ is the transmitted signal energy per bit.
  - A pair of sinusoidal waves that differ only in a relative phase-shift of 180 degrees are referred to as *antipodal signals*.

# Memoryless Modulation Methods

- Phase-modulated signals (Binary Phase-Shift Keying)
  - To ensure that each transmitted bit contains an integral number of cycles of the carrier wave, the carrier frequency $f_c$ is chosen equal to $n_c/T_b$ for some fixed integer $n_c$.
  - In the case of binary PSK, there is only one basis function of unit energy:

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), \qquad 0 \le t < T_b$$

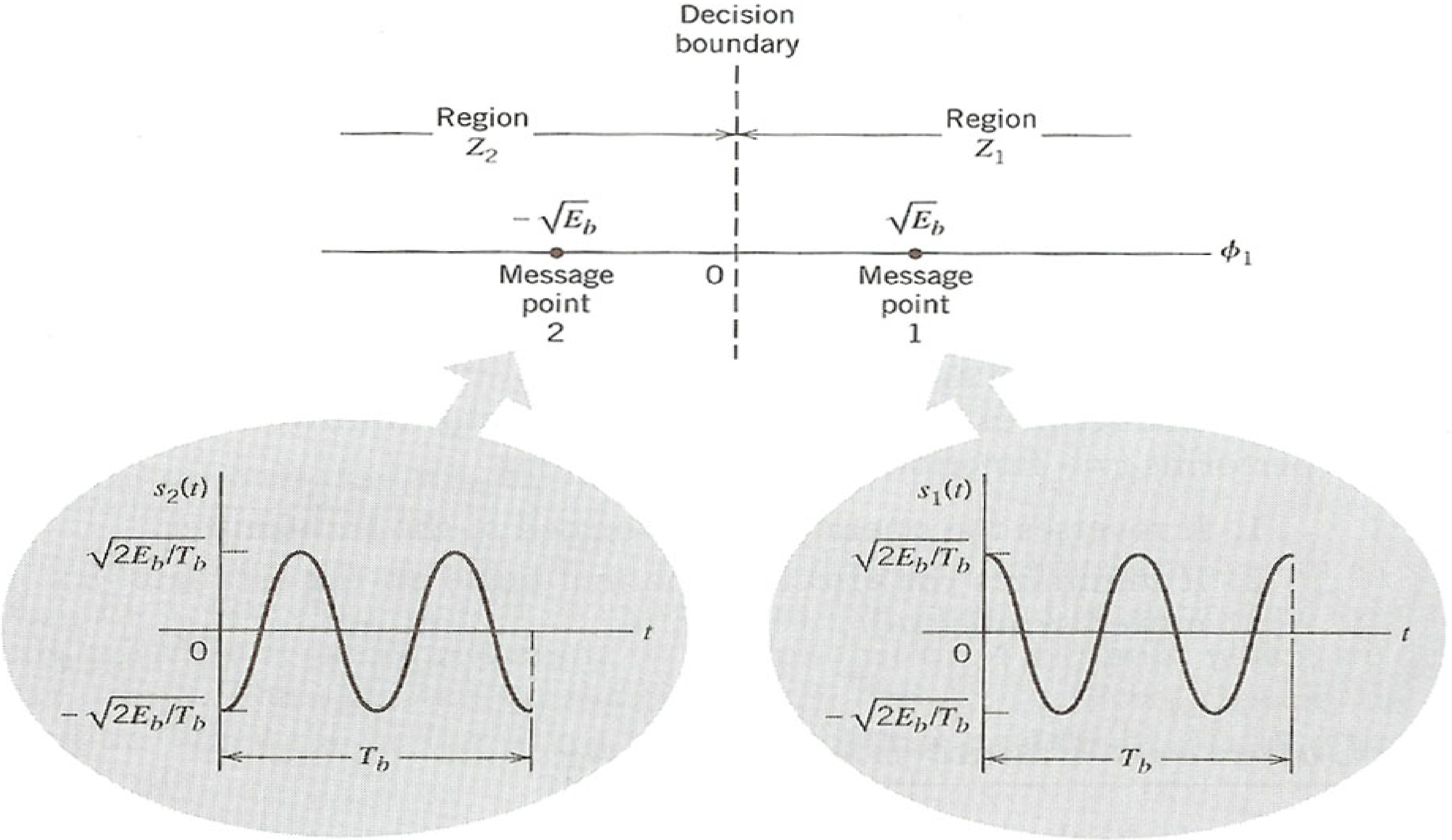
$$s_1(t) = \sqrt{E_b}\phi_1(t),\ s_2(t) = -\sqrt{E_b}\phi_1(t),\ 0 \le t < T_b$$

  - The coordinates of the message points are:

$$s_{11} = \int_0^{T_b} s_1(t)\phi_1(t)\,dt = +\sqrt{E_b} \qquad s_{21} = \int_0^{T_b} s_2(t)\phi_1(t)\,dt = -\sqrt{E_b}$$

# Memoryless Modulation Methods

- Phase-modulated signals (Binary Phase-Shift Keying)



# Memoryless Modulation Methods

- Phase-modulated signals (Quadriphase-Shift Keying)
  - Quadriphase-Shift Keying (QPSK)

$$s_i(t)=\begin{cases}\sqrt{\frac{2E}{T}}\cos\left[2\pi f_c t+(2i-1)\frac{\pi}{4}\right], & 0\le t\le T\\ 0, & \text{elsewhere}\end{cases}$$

where $i$ = 1, 2, 3, 4; $E$ is the transmitted signal energy per symbol, and $T$ is the symbol duration.

  - Equivalently

$$s_i(t)=\sqrt{\frac{2E}{T}}\cos\left[(2i-1)\frac{\pi}{4}\right]\cos(2\pi \mathrm{f}_c t)-\sqrt{\frac{2E}{T}}\sin\left[(2i-1)\frac{\pi}{4}\right]\sin(2\pi \mathrm{f}_c t)$$

# Memoryless Modulation Methods

- Phase-modulated signals (Quadriphase-Shift Keying)
  - Defined a pair of quadrature carriers:

$$\phi_1(t)=\sqrt{\frac{2}{T}}\cos(2\pi f_c t), \qquad 0\le t\le T$$

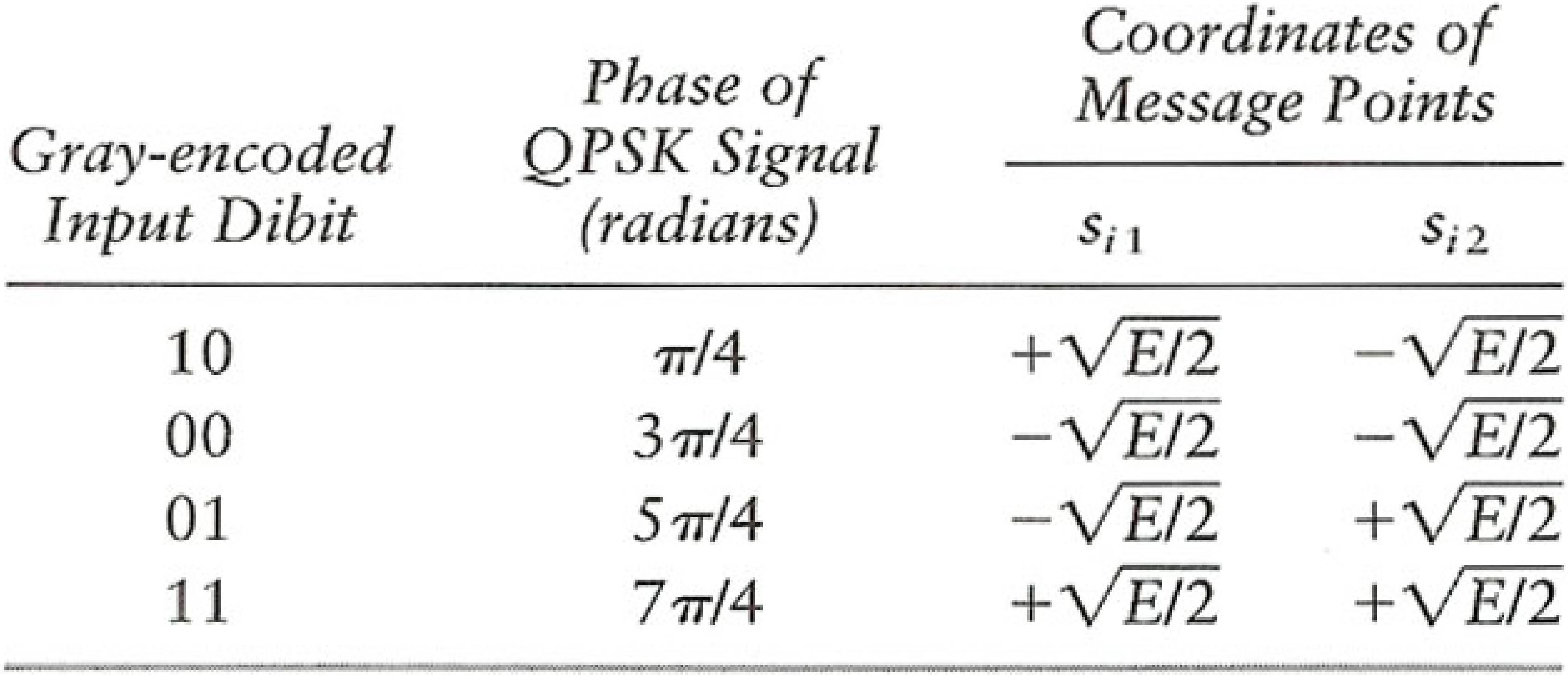
$$\phi_2(t)=\sqrt{\frac{2}{T}}\sin(2\pi f_c t), \qquad 0\le t\le T$$

  - There are four message points, and the associated signal vectors are defined by

$$s_i(t)=\begin{bmatrix}\sqrt{E}\cos\left((2i-1)\frac{\pi}{4}\right)\\ -\sqrt{E}\sin\left((2i-1)\frac{\pi}{4}\right)\end{bmatrix}, \qquad i=1,2,3,4$$

# Memoryless Modulation Methods

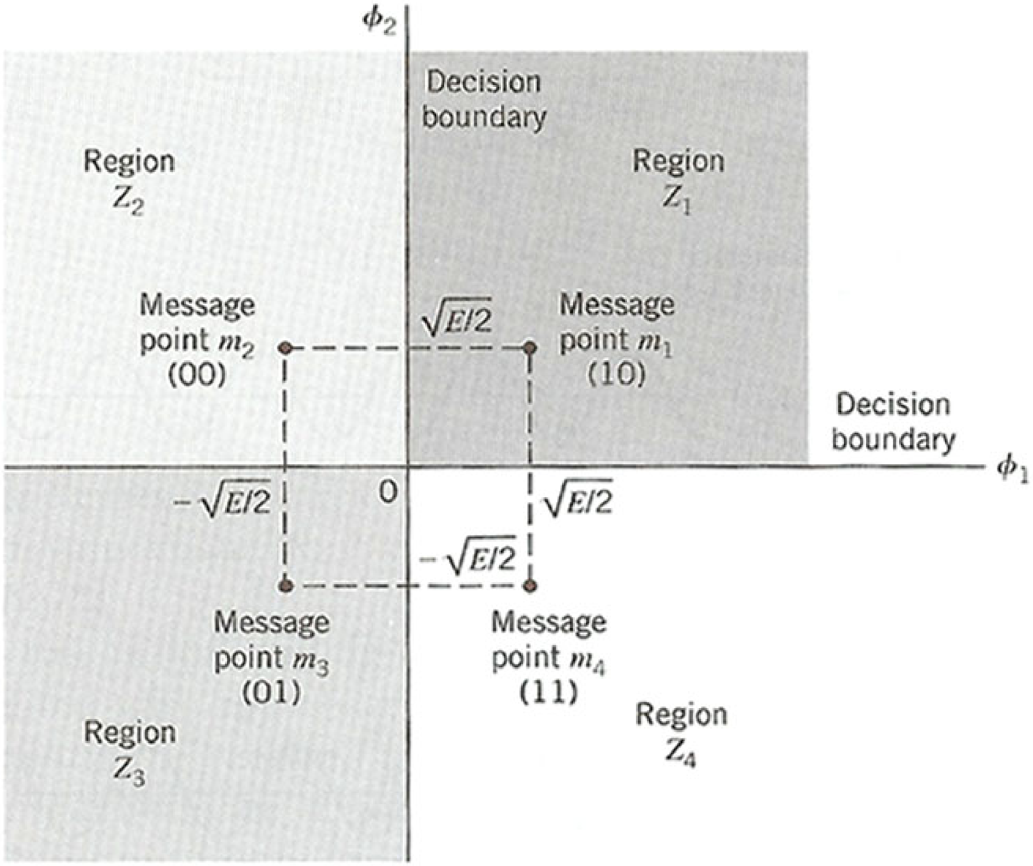
- Phase-modulated signals (Quadriphase-Shift Keying)
  - Each possible value of the phase corresponds to a unique dibit.
  - For example, we may choose the Gray coding.



| Gray-encoded Input Dibit | Phase of QPSK Signal (radians) | Coordinates of Message Points | |
|---|---|---|---|
| | | $s_{i1}$ | $s_{i2}$ |
| 10 | $\pi/4$ | $+\sqrt{E/2}$ | $-\sqrt{E/2}$ |
| 00 | $3\pi/4$ | $-\sqrt{E/2}$ | $-\sqrt{E/2}$ |
| 01 | $5\pi/4$ | $-\sqrt{E/2}$ | $+\sqrt{E/2}$ |
| 11 | $7\pi/4$ | $+\sqrt{E/2}$ | $+\sqrt{E/2}$ |

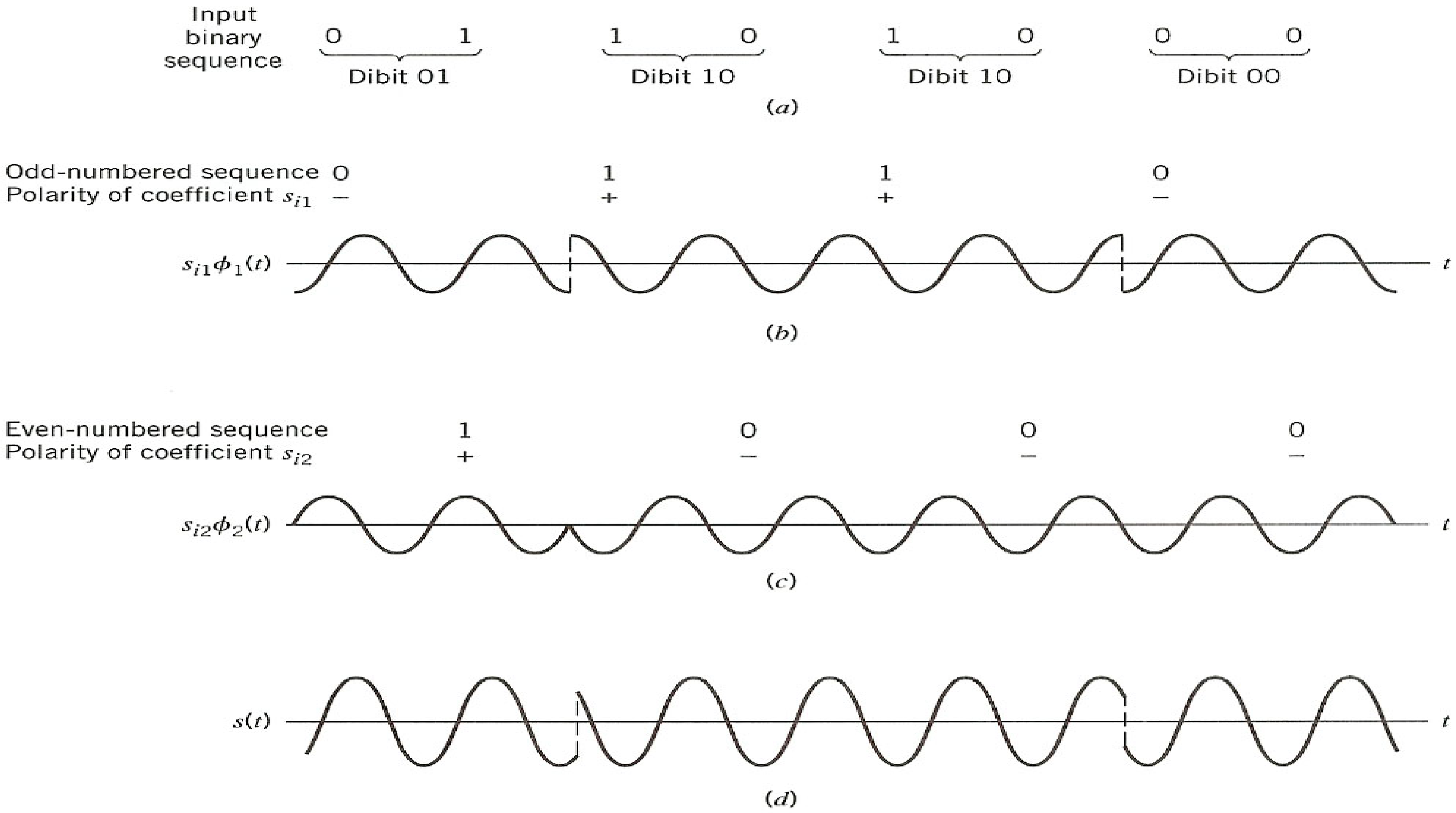
# Memoryless Modulation Methods

- Phase-modulated signals (Quadriphase-Shift Keying)
  - Signal space diagram of coherent QPSK system



# Memoryless Modulation Methods

- Phase-modulated signals (Quadriphase-Shift Keying)



# Memoryless Modulation Methods

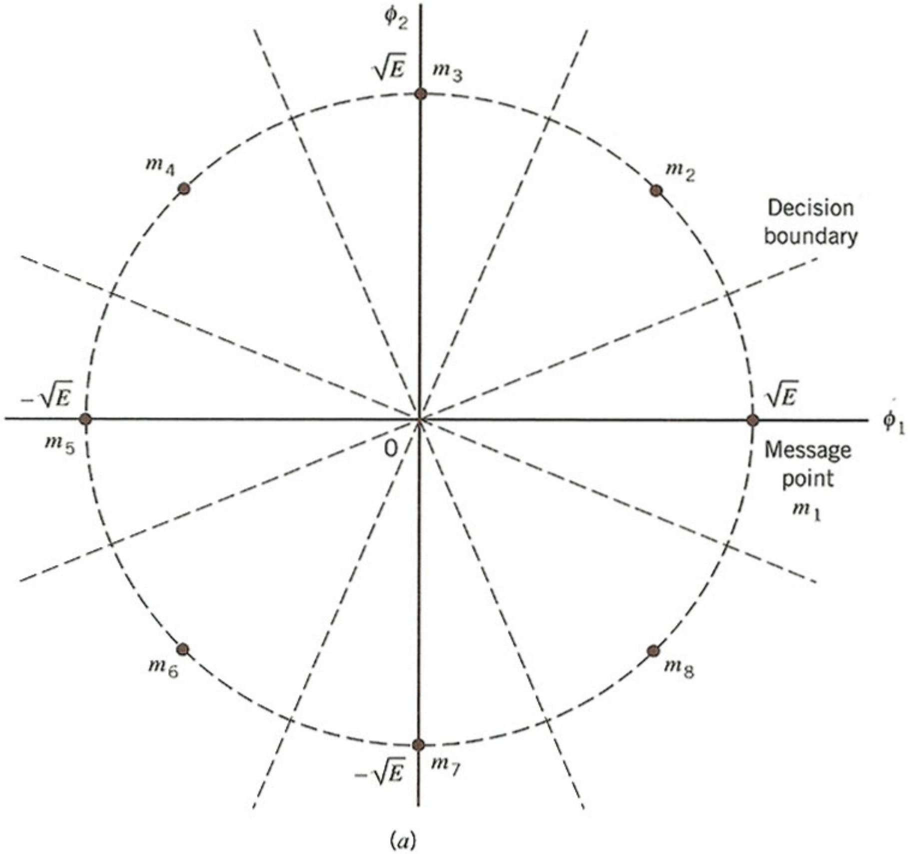
- Phase-modulated signals (*M*-ary PSK)
  - The *M* signal waveforms are represented as:

$$
\begin{aligned}
&s_m(t) \\
&= \operatorname{Re}\left[ g(t) e^{j2\pi(m-1)/M} e^{j2\pi f_c t} \right], \quad m = 1, 2, \ldots, M, \quad 0 \le t \le T \\
&= g(t) \cos\left[ 2\pi f_c t + \frac{2\pi}{M}(m-1) \right] \\
&= g(t) \cos \frac{2\pi}{M}(m-1) \cos 2\pi f_c t - g(t) \sin \frac{2\pi}{M}(m-1) \sin 2\pi f_c t
\end{aligned}
$$

  - *Digital phase modulation* is usually called *phase-shift keying (PSK)*.

# Memoryless Modulation Methods

- Phase-modulated signals (*M*-ary PSK)
  - Signal space diagram for octaphase shift keying (i.e., *M*=8)



(*a*)

# Memoryless Modulation Methods

- Phase-modulated signals (*M*-ary PSK)
  - The signal waveforms have equal energy:

$$\varepsilon = \int_0^T s_m^2(t)\,dt = \frac{1}{2}\int_0^T g^2(t)\,dt = \frac{1}{2}\varepsilon_g$$

  - The signal waveforms may be represented as a linear combination of two orthonormal signal waveforms:

$$s_m(t) = s_{m1} f_1(t) + s_{m2} f_2(t)$$

$$f_1(t) = \sqrt{\frac{2}{\varepsilon_g}}\, g(t)\cos 2\pi f_c t \quad \text{and} \quad f_2(t) = -\sqrt{\frac{2}{\varepsilon_g}}\, g(t)\sin 2\pi f_c t$$

  - The two-dimensional vectors $s_m=[s_{m1}\ s_{m2}]$ are given by:

$$s_m = \left[\sqrt{\frac{2}{\varepsilon_g}}\cos\frac{2\pi}{M}(m-1) \quad \sqrt{\frac{2}{\varepsilon_g}}\sin\frac{2\pi}{M}(m-1)\right], \quad m = 1, 2, \ldots, M$$

# Memoryless Modulation Methods

- Phase-modulated signals (*M*-ary PSK)
  - The Euclidean distance between signal points is:

$$d_{mn}^{(e)} = \left\| s_m - s_n \right\|$$

$$= \sqrt{\left(s_{m1} - s_{n1}\right)^2 + \left(s_{m2} - s_{n2}\right)^2} = \left\{ \varepsilon_g \left[ 1 - \cos\frac{2\pi}{M}(m-n) \right] \right\}^{1/2}$$

  - The minimum Euclidean distance corresponds to the case in which |*m*-*n*|=1, i.e., adjacent signal phases.

$$d_{\min}^{(e)} = \sqrt{\varepsilon_g \left( 1 - \cos\frac{2\pi}{M} \right)}$$

# Memoryless Modulation Methods

- Phase-modulated signals (Offset QPSK)
  - The carrier phase changes by ±180 degrees whenever both the in-phase and quadrature components of the QPSK signal changes sign.
  - This can result in problems for power amplifiers.
  - The problem may be reduced by using offset QPSK.
  - In offset QPSK, the bit stream responsible for generating the quadrature component is delayed (i.e. offset) by half a symbol interval with respect to the bit stream responsible for generating the in-phase component.

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t), \qquad 0 \le t \le T$$

# Memoryless Modulation Methods

- Phase-modulated signals (Offset QPSK)
  - The two basis functions of offset QPSK are defined by

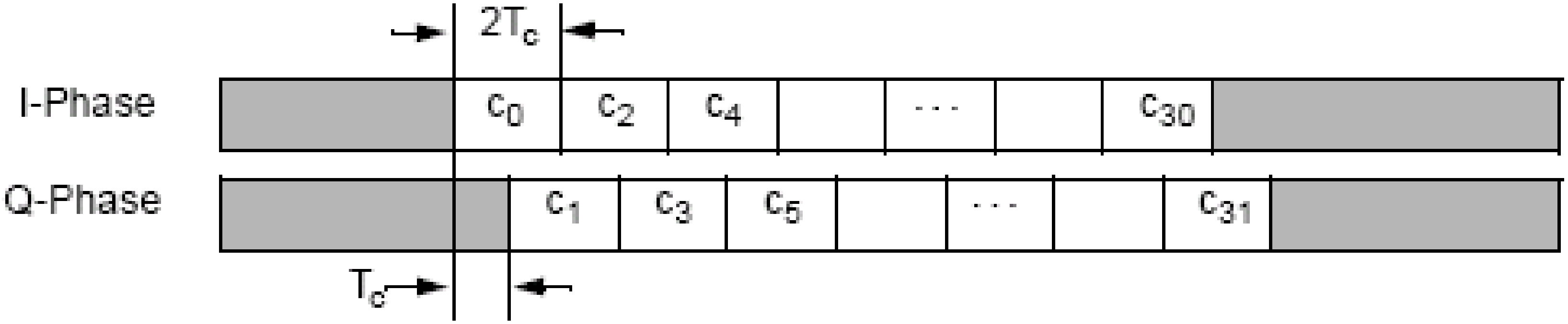
$$\phi_1(t)=\sqrt{\frac{2}{T}}\cos(2\pi f_c t), \qquad 0\le t\le T$$

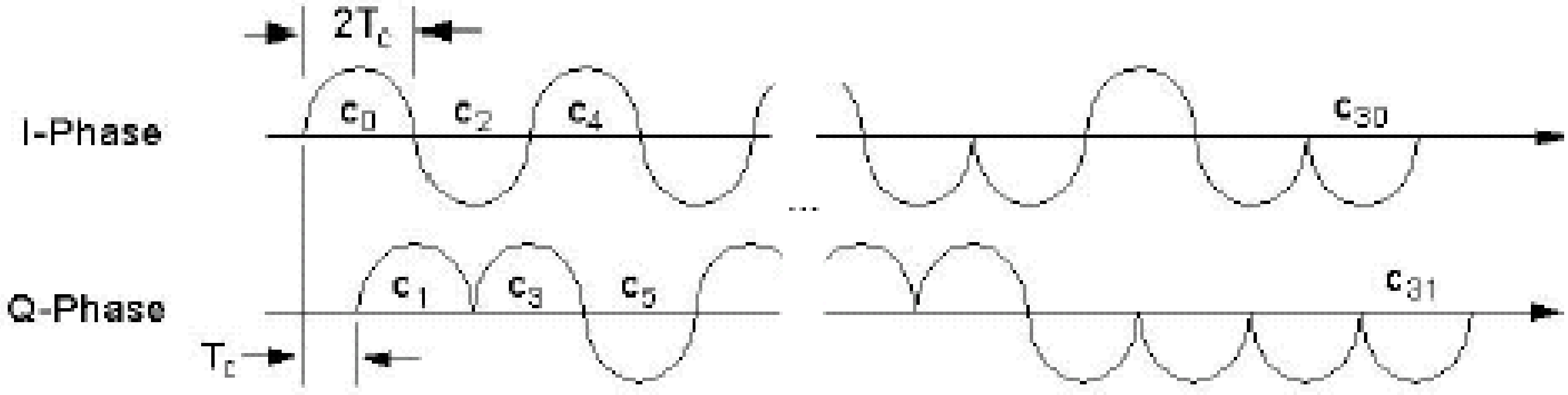
$$\phi_2(t)=\sqrt{\frac{2}{T}}\sin(2\pi f_c t), \qquad \frac{T}{2}\le t\le \frac{3T}{2}$$

  - The phase transitions likely to occur in offset QPSK are confined to ±90 degrees.
  - However, ±90 degrees phase transitions in offset QPSK occur twice as frequently.

# Memoryless Modulation Methods

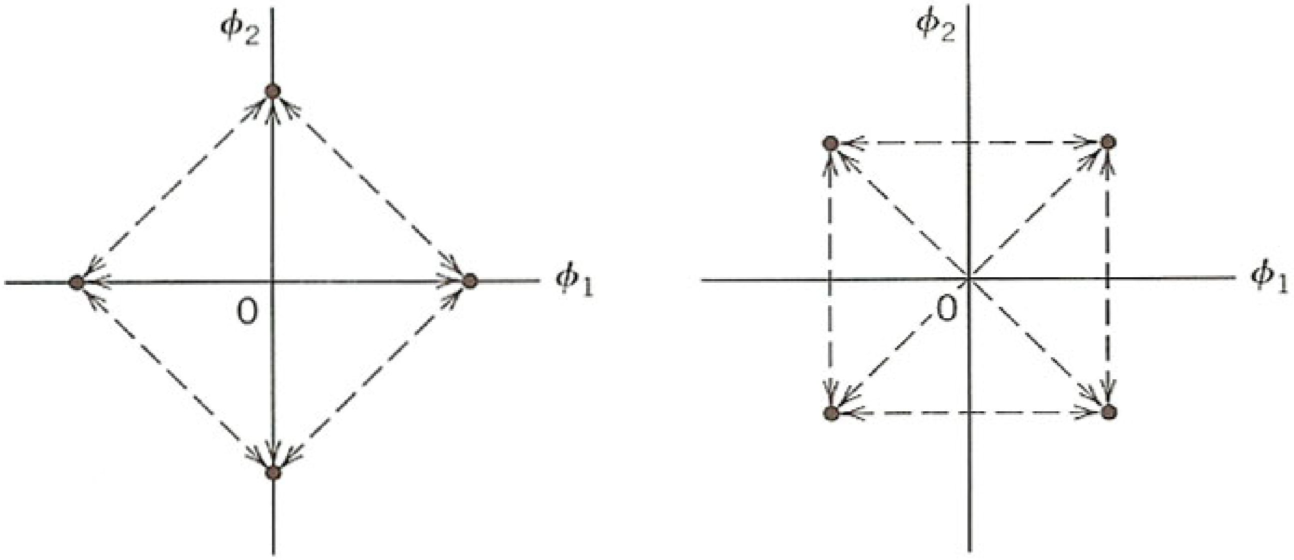
- Phase-modulated signals (Offset QPSK)





# Memoryless Modulation Methods

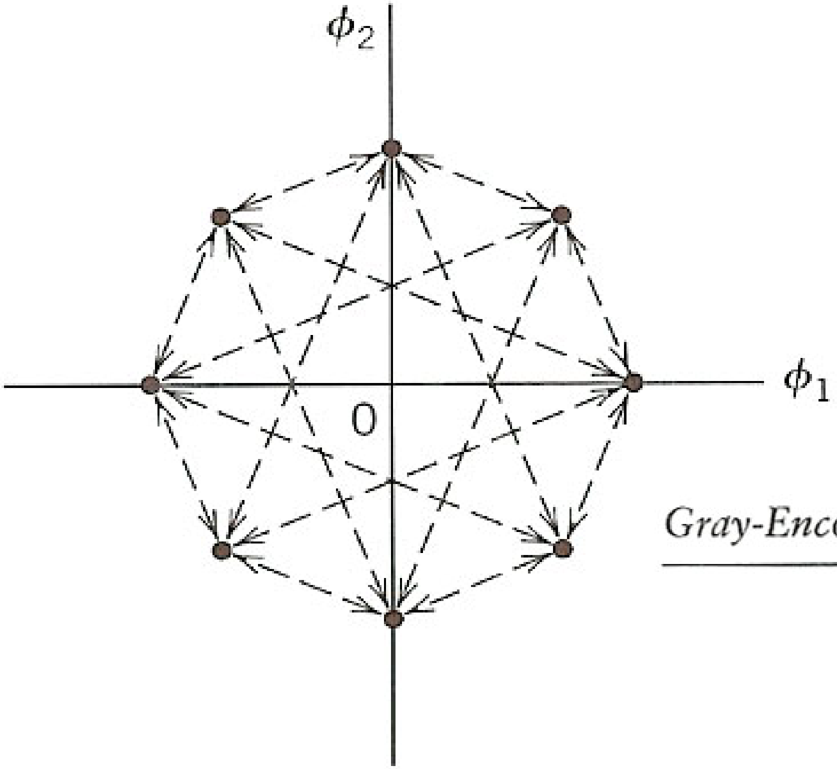
- Phase-modulated signals (π/4-Shifted QPSK)
  - The carrier phase used for the transmission of successive symbols is alternately picked from one of the two QPSK constellations in the following figure and then the other.



(a) (b)

# Memoryless Modulation Methods

- Phase-modulated signals (π/4-Shifted QPSK)
  - It follows that a π/4-shifted QPSK signal may reside in any one of eight possible phase states:



| *Gray-Encoded Input Dibit* | *Phase Change, Δθ (radians)* |
|---|---|
| 00 | $\pi/4$ |
| 01 | $3\pi/4$ |
| 11 | $-3\pi/4$ |
| 10 | $-\pi/4$ |

# Memoryless Modulation Methods

- Phase-modulated signals (π/4-Shifted QPSK)
  - Attractive features of the π/4-shifted QPSK scheme
    - The phase transitions from one symbol to the next are restricted to ±π/4 and ±3π/4.
    - Envelope variations due to filtering are significantly reduced.
    - π/4-shifted QPSK signals can be *noncoherently detected*, thereby considerably simplifying the receiver design.
    - Like QPSK signals, π/4-shifted QPSK can be differently encoded, in which case we should really speak of π/4-shifted DQPSK .
    - π/4-DQPSK is adopted in IS-54/136.

# Memoryless Modulation Methods

- Quadrature amplitude modulation (QAM)
  - *Quadrature PAM* or *QAM*: The bandwidth efficiency of PAM/SSB can also be obtained by simultaneously impressing two separate *k*-bit symbols from the information sequence $\{a_n\}$ on two quadrature carriers $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$.
  - The signal waveforms may be expressed as:

$$s_m(t) = \mathrm{Re}\left[\left(A_{mc} + jA_{ms}\right) g(t) e^{j2\pi f_c t}\right], \; m = 1, 2, \ldots, M, \; 0 \le t \le T$$

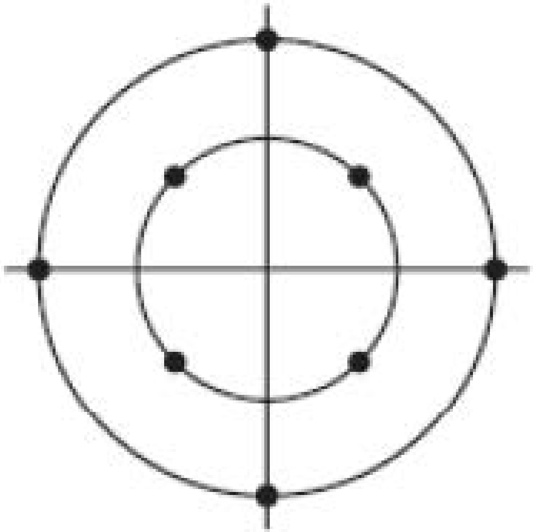
$$= A_{mc} g(t) \cos 2\pi f_c t - A_{ms} g(t) \sin 2\pi f_c t$$

$$s_m(t) = \mathrm{Re}\left[V_m e^{j\theta_m} g(t) e^{j2\pi f_c t}\right] = V_m g(t) \cos\left(2\pi f_c t + \theta_m\right)$$

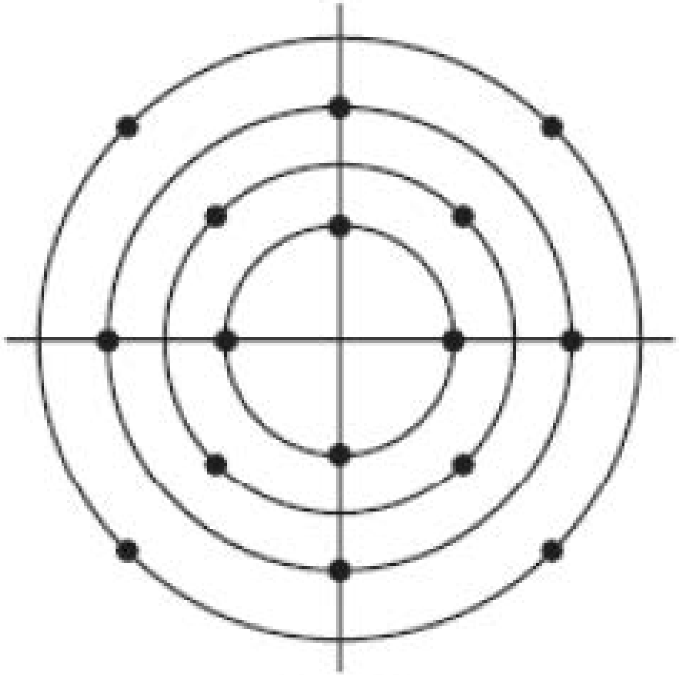
$$V_m = \sqrt{A_{mc}^2 + A_{ms}^2} \quad \text{and} \quad \theta_m = \tan^{-1}\left(A_{ms} / A_{mc}\right)$$

# Memoryless Modulation Methods

- Quadrature amplitude modulation (QAM) (cont.)
  - We may select a combination of $M_1$-level PAM and $M_2$-phase PSK to construct an $M=M_1M_2$ combined PAM-PSK signal constellation.



$M=8$



$M=16$

# Memoryless Modulation Methods

- Quadrature amplitude modulation (QAM) (cont.)
  - As in the case of PSK signals, the QAM signal waveforms may be represented as a linear combination of two orthonormal signal waveforms $f_1(t)$ and $f_2(t)$:

$$s_m(t) = s_{m1} f_1(t) + s_{m2} f_2(t)$$

$$f_1(t) = \sqrt{\frac{2}{\varepsilon_g}} g(t) \cos 2\pi f_c t \quad \text{and} \quad f_2(t) = -\sqrt{\frac{2}{\varepsilon_g}} g(t) \sin 2\pi f_c t$$

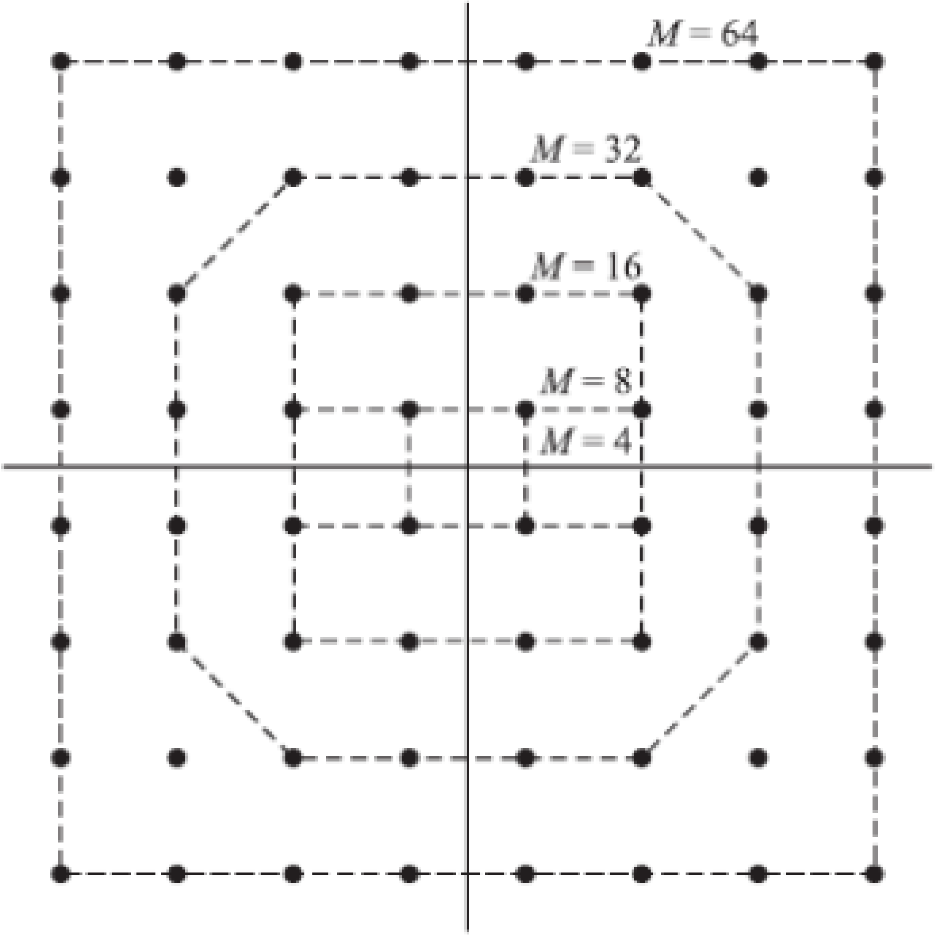
$$\mathbf{s}_m = \begin{bmatrix} s_{m1} & s_{m2} \end{bmatrix} = \begin{bmatrix} A_{mc}\sqrt{\frac{1}{2}\varepsilon_g} & A_{ms}\sqrt{\frac{1}{2}\varepsilon_g} \end{bmatrix}$$

  - The Euclidean distance between any pair of signal vectors is:

$$d_{mn}^{(e)} = \left\| \mathbf{s}_m - \mathbf{s}_n \right\| = \sqrt{\frac{1}{2}\varepsilon_g \left[ \left(A_{mc} - A_{nc}\right)^2 + \left(A_{ms} - A_{ns}\right)^2 \right]}$$

# Memoryless Modulation Methods

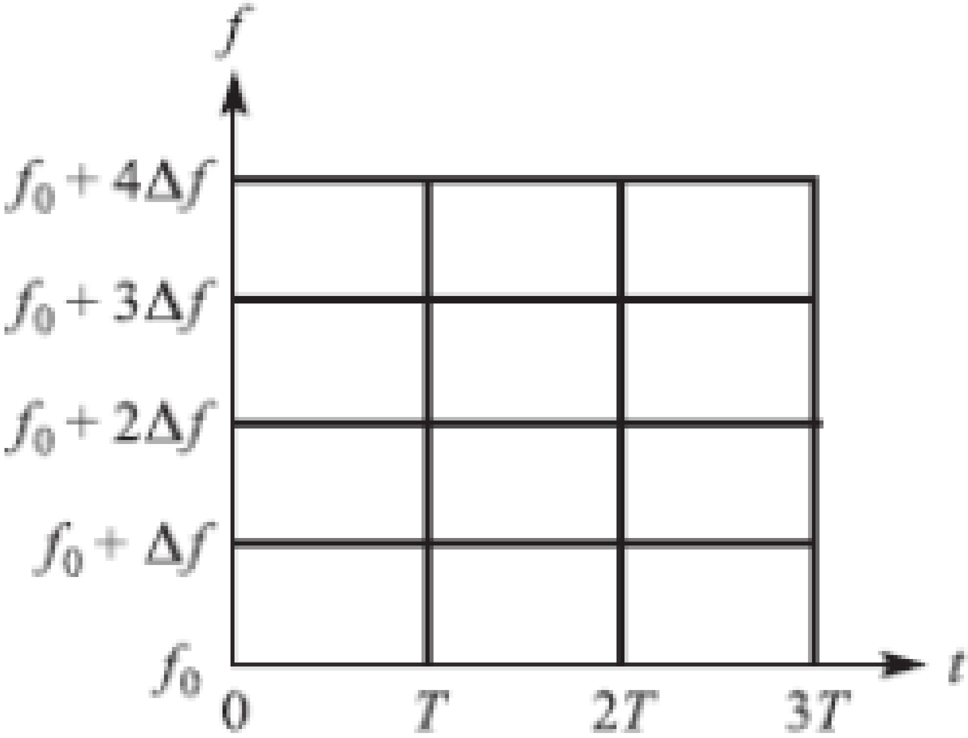
- Quadrature amplitude modulation (QAM) (cont.)
  - Several signal space diagrams for *rectangular QAM*:



$$d_{\min}^{(e)} = d\sqrt{2\varepsilon_g}$$

# Memoryless Modulation Methods

- Multidimensional signals:
  - We may use either the time domain or the frequency domain or both in order to increase the number of dimensions.
  - Subdivision of time and frequency axes into distinct slots:



# Memoryless Modulation Methods

- Orthogonal multidimensional signals
  - Consider the construction of *M* equal-energy orthogonal signal waveforms that differ in frequency:

$$s_m(t) = \mathrm{Re}\left[ s_{lm}(t) e^{j2\pi f_c t} \right], \quad m = 1, 2, \ldots, M, \quad 0 \le t \le T$$

$$= \sqrt{\frac{2\varepsilon}{T}} \cos\left[ 2\pi f_c t + 2\pi m \Delta f \ t \right]$$

$$s_{lm}(t) = \sqrt{\frac{2\varepsilon}{T}} e^{j2\pi m \Delta f \ t}, \quad m = 1, 2, \ldots, M, \quad 0 \le t \le T$$

  - This type of frequency modulation is called *frequency-shift keying (FSK)*.

# Memoryless Modulation Methods

- Orthogonal multidimensional signals (cont.)
  - These waveforms have equal cross-correlation coefficients:

$$\rho_{km} = \frac{2\varepsilon/T}{2\varepsilon}\int_0^T e^{j2\pi(m-k)\Delta ft}dt = \frac{\sin \pi T(m-k)\Delta f}{\pi T(m-k)\Delta f}e^{j\pi T(m-k)\Delta f}$$

$$\rho_{km} = \frac{1}{2\sqrt{\varepsilon_m\varepsilon_k}}\int_{-\infty}^{\infty} s_{lm}^*(t)s_{lk}(t)dt$$

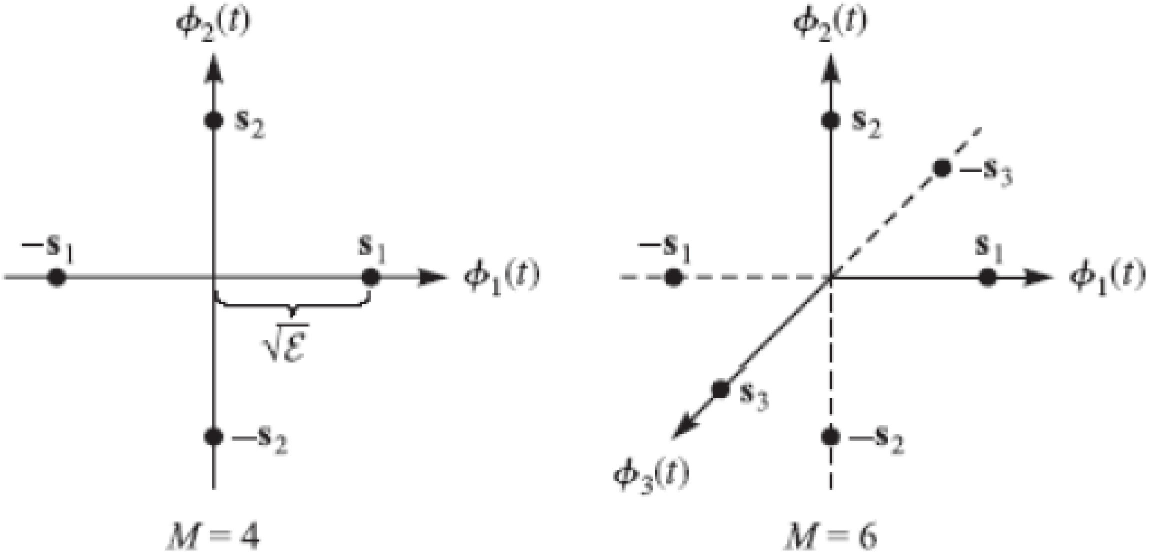
$$\rho_r \equiv \mathrm{Re}(\rho_{km}) = \frac{\sin[\pi T(m-k)\Delta f]}{\pi T(m-k)\Delta f}\cos[\pi T(m-k)\Delta f]$$

$$= \frac{\sin[2\pi T(m-k)\Delta f]}{2\pi T(m-k)\Delta f}$$

  - Note that Re($\rho_{km}$)=0 when $\Delta f$=1/2$T$ and $m \neq k$. => Orthogonal
  - The minimum frequency separation $\Delta f$ that guarantees orthogonality is $\Delta f$=1/2$T$.

# Memoryless Modulation Methods

- Biorthogonal signals
  - A set of *M* biorthogonal signals can be constructed from *M*/2 orthogonal signals by simply including the negatives of the orthogonal signals.
  - The correlation between any pair of waveforms is either $\rho_r$=-1 or 0.



Signal space diagrams for *M*=4 and *M*=6 biorthogonal signals.

# Memoryless Modulation Methods

- Simplex signals
  - For a set of $M$ orthogonal waveforms $\{s_m(t)\}$ or their vector representation $\{s_m\}$ with mean of:

$$\bar{s} = \frac{1}{M}\sum_{m=1}^{M} s_m$$

*Simplex signals* are obtained by translating the origin of the $m$ orthogonal signals to the point $\bar{s}$ .

$$s_m^{'} = s_m - \bar{s}, \quad m = 1, 2, ..., M$$

  - The energy per waveform is:

$$\left\| s_m^{'} \right\|^2 = \left\| s_m - \bar{s} \right\|^2 = \varepsilon - \frac{2}{M}\varepsilon + \frac{1}{M}\varepsilon = \varepsilon\left(1 - \frac{1}{M}\right)$$

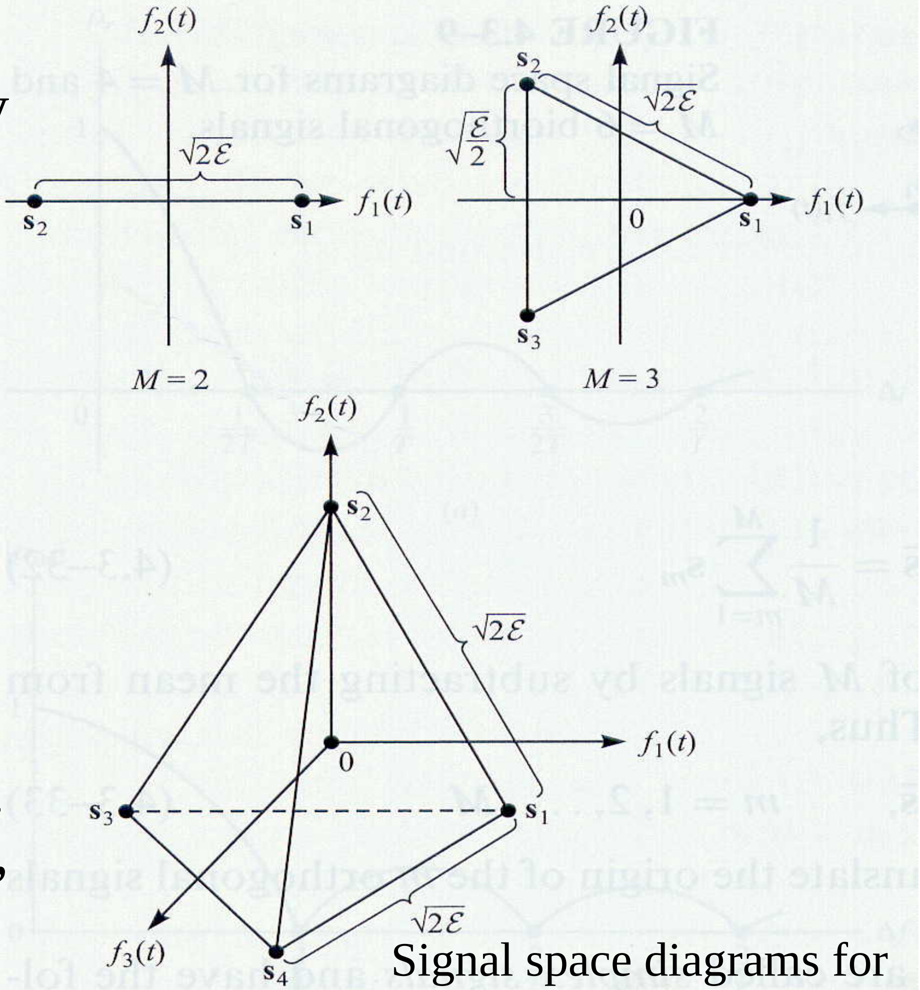
# Memoryless Modulation Methods

- Simplex signals (cont.)
  - The cross correlation of any pair of signals is:

$$\mathrm{Re}\left(\rho_{mn}\right) = \frac{s'_m \cdot s'_n}{\left\|s'_m\right\|\left\|s'_n\right\|} = \frac{-1/M}{1-1/M}$$

$$= -\frac{1}{M-1}$$

  - The set of simplex waveforms is *equally correlated* and requires less energy, by the factor 1-1/$M$, than the set of orthogonal waveforms.



Signal space diagrams for $M$-ary simplex signals.

# Non-linear Modulation Methods with Memory

- Introduction
  - In this section, we consider a class of digital modulation methods in which the phase of the signal is constrained to be continuous.
  - This constraint results in a phase or frequency modulator that has memory.
  - The modulation method is also non-linear.
- Continuous-phase FSK (CPFSK)
  - A conventional FSK signal is generated by shifting the carrier by an amount $f_n = \frac{1}{2}\Delta f \ I_n, \ I_n = \pm 1, \pm 3, \ldots, \pm(M-1)$, to reflect the digital information that is being transmitted.
  - This type (conventional type) of FSK signal is memoryless.

# Non-linear Modulation Methods with Memory

- Continuous-phase FSK (CPFSK) (cont.)
  - The switching from one frequency to another may be accomplished by having $M=2^k$ separate oscillators tuned to the desired frequencies and selecting one of the $M$ frequencies according to the particular $k$-bit symbol that is to be transmitted in a signal interval of duration $T=k/R$ seconds.
  - The reasons why we have CPFSK: (or the defects of conventional FSK)
    - Such abrupt switching from one oscillator output to another in successive signaling intervals results in relatively large spectral side lobes outside of the main spectral band of the signal.
    - Consequently, this method requires a large frequency band for transmission of the signal.

# Non-linear Modulation Methods with Memory

- Continuous-phase FSK (CPFSK) (cont.)
  - Solution:
    - To avoid the use of signals having large spectral side lobes, the information-bearing signal frequency modulates a single carrier whose frequency is changed continuously.
    - The resulting frequency-modulated signal is phase-continuous and, hence, it is called *continuous-phase FSK (CPFSK)* .
  - This type (continuous-phase type) of FSK signal has memory because the phase of the carrier is constrained to be continuous.

# Non-linear Modulation Methods with Memory

- Continuous-phase FSK (CPFSK) (cont.)
  - In order to represent a CPFSK signal, we begin with a PAM signal:

$$d(t) = \sum_n I_n g(t - nT)$$

  - $d(t)$ is used to frequency-modulate the carrier.
  - $\{I_n\}$ denotes the sequence of amplitudes obtained by mapping $k$-bit blocks of binary digits from the information sequence $\{a_n\}$ into the amplitude levels $\pm 1, \pm 3, \cdots, \pm(M-1)$.
  - $g(t)$ is a rectangular pulse of amplitude $1/2T$ and duration $T$ seconds.

# Non-linear Modulation Methods with Memory

- Continuous-phase FSK (CPFSK) (cont.)
  - Equivalent low-pass waveform $v(t)$ is expressed as

$$v(t) = \sqrt{\frac{2\varepsilon}{T}} \exp\left\{ j\left[ 4\pi T f_d \int_{-\infty}^{t} d(\tau) d\tau + \phi_0 \right]\right\}$$

    - $f_d$ is the *peak frequency deviation*, $\varphi_0$ is the initial phase of the carrier.
  - The carrier-modulated signal may be expressed as

$$s(t) = \sqrt{\frac{2\varepsilon}{T}} \cos\left[ 2\pi f_c t + \phi(t;\mathrm{I}) + \phi_0 \right]$$

where $\varphi(t;\mathrm{I})$ represents the time-varying phase of the carrier.

# Non-linear Modulation Methods with Memory

- Continuous-phase FSK (CPFSK) (cont.)
  - $$\phi(t;\mathrm{I}) = 4\pi T f_d \int_{-\infty}^{t} d(\tau)\,d\tau = 4\pi T f_d \int_{-\infty}^{t} \left[ \sum_n I_n g(\tau - nT) \right] d\tau$$

    $$= 2\pi f_d T \sum_{k=-\infty}^{n-1} I_k + 2\pi f_d (t - nT) I_n \qquad \boxed{nT \le t \le (n+1)T}$$

    $$= \theta_n + 2\pi h I_n q(t' - nT)$$

    $$\boxed{\theta_n = \pi h \sum_{k=-\infty}^{n-1} I_k} \qquad \boxed{h = 2 f_d T} \qquad \boxed{q(t') = \begin{cases} 0 & (t' < 0) \\ t'/2T & (0 \le t' \le T) \\ 1/2 & (t' > T) \end{cases}}$$
  - Note that, although *d*(*t*) contains discontinuities, the integral of *d*(*t*) is continuous. Hence, we have a continuous-phase signal.
  - $\theta_n$ represents the accumulation (memory) of all symbols up to time *nT* .
  - Parameter *h* is called the *modulation index*.

# Non-linear Modulation Methods with Memory

- Continuous-phase modulation (CPM)
  - CPFSK becomes a special case of a general class of *continuous-phase modulated* (CPM) signals in which the carrier phase is

$$\phi(t;I) = 2\pi \sum_{k=-\infty}^{n} I_k h_k q(t-kT), \qquad nT \le t \le (n+1)T$$

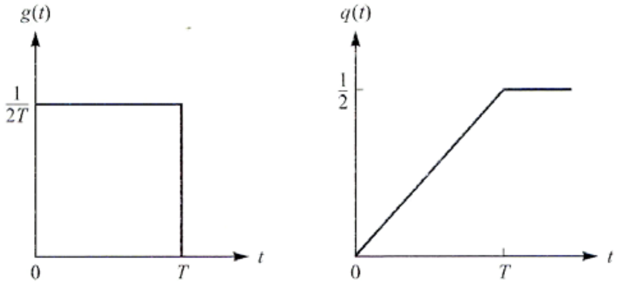
    - when $h_k = h$ for all $k$, the modulation index is fixed for all symbols.
    - when $h_k$ varies from one symbol to another, the CPM signal is called *multi-h*. (In such a case, the $\{h_k\}$ are made to vary in a cyclic manner through a set of indices.)
  - The waveform $q(t)$ may be represented in general as the integral of some pulse $g(t)$, i.e.,

$$q(t) = \int_0^t g(\tau)\,d\tau$$

# Non-linear Modulation Methods with Memory

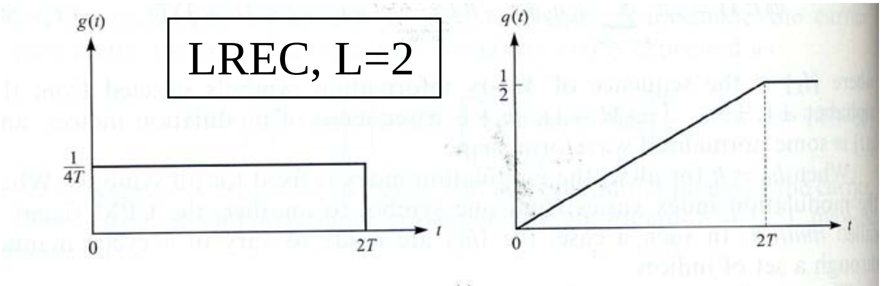
- Continuous-phase modulation (CPM) (cont.)
  - If $g(t)=0$ for $t>T$, the CPM signal is called *full response CPM*. (Fig a. b.)
  - If $g(t)\neq 0$ for $t>T$, the modulated signal is called *partial response CPM*. (Fig c. d.)

LREC, L=1, results in CPFSK



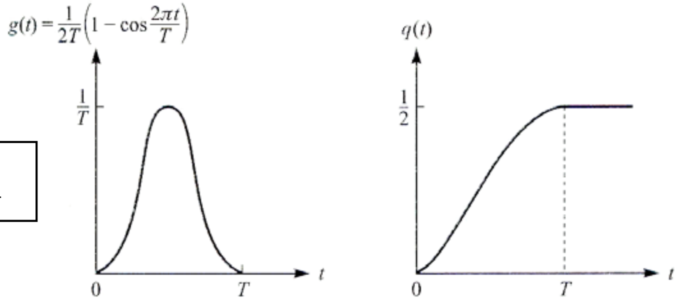
(a)

LREC, L=2



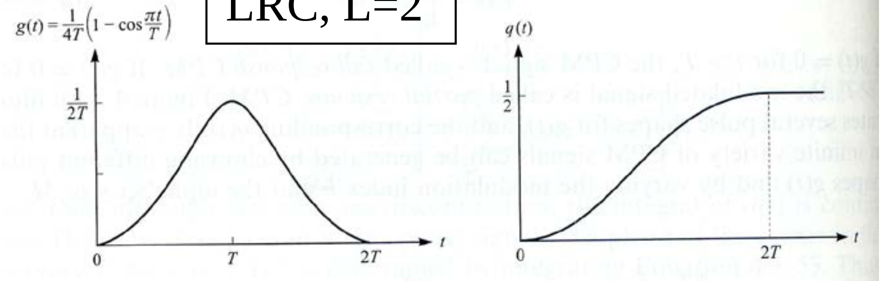
(c)

LRC, L=1



(b)

LRC, L=2



(d)

# Non-linear Modulation Methods with Memory

- Continuous-phase modulation (CPM) (cont.)
  - The CPM signal has memory that is introduced through the phase continuity.
  - For $L>1$, additional memory is introduced in the CPM signal by the pulse $g(t)$.
  - Three popular pulse shapes are given in the following table.
    - LREC denotes a rectangular pulse of duration $LT$.
    - LRC denotes a raised cosine pulse of duration $LT$.
    - *Gaussian minimum-shift keying* (GMSK) pulse with bandwidth parameter $B$, which represents the -3 dB bandwidth of the Gaussian pulse.

# Non-linear Modulation Methods with Memory

- Continuous-phase modulation (CPM) (cont.)
  - Some commonly used CPM pulse shapes
  - LREC
$$g(t)=\begin{cases}\dfrac{1}{2LT} & (0\le 1\le LT)\\ 0 & (\text{otherwise})\end{cases}$$
  - LRC
$$g(t)=\begin{cases}\dfrac{1}{2LT}\left(1-\cos\dfrac{2\pi t}{LT}\right) & (0\le 1\le LT)\\ 0 & (\text{otherwise})\end{cases}$$
  - GMSK
$$g(t)=\left\{Q\left[2\pi B\left(t-\frac{T}{2}\right)(\ln 2)^{1/2}\right]-Q\left[2\pi B\left(t+\frac{T}{2}\right)(\ln 2)^{1/2}\right]\right\}$$
$$Q(t)=\int_t^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dt$$

# Non-linear Modulation Methods with Memory

- Minimum-shift keying (MSK).
  - MSK is a special form of binary CPFSK (and, therefore, CPM) in which the modulation index $\underline{h=1/2}$.
  - The phase of the carrier in the interval $nT \le t \le (n+1)T$ is

$$\phi(t;\mathrm{I}) = \frac{1}{2}\pi \sum_{k=-\infty}^{n-1} I_k + \pi I_n q(t-nT)$$

$$= \theta_n + \frac{1}{2}\pi I_n \left( \frac{t-nT}{T} \right), \qquad nT \le t \le (n+1)T$$

  - The modulated carrier signal is

$$s(t) = A\cos\left[ 2\pi f_c t + \theta_n + \frac{1}{2}\pi I_n \left( \frac{t-nT}{T} \right) \right]$$

$$= A\cos\left[ 2\pi \left( f_c + \frac{1}{4T} I_n \right) t - \frac{1}{2} n\pi I_n + \theta_n \right], \qquad nT \le t \le (n+1)T$$

# Non-linear Modulation Methods with Memory

- Minimum-shift keying (MSK) (cont.)
  - The expression indicates that the binary CPFSK signal can be expressed as a sinusoid having one of two possible frequencies in the interval $nT \le t \le (n+1)T$. If we define these frequencies as

$$f_1 = f_c + \frac{1}{4T}$$

$$f_2 = f_c - \frac{1}{4T}$$

  - Then the binary CPFSK signal may be written in the form

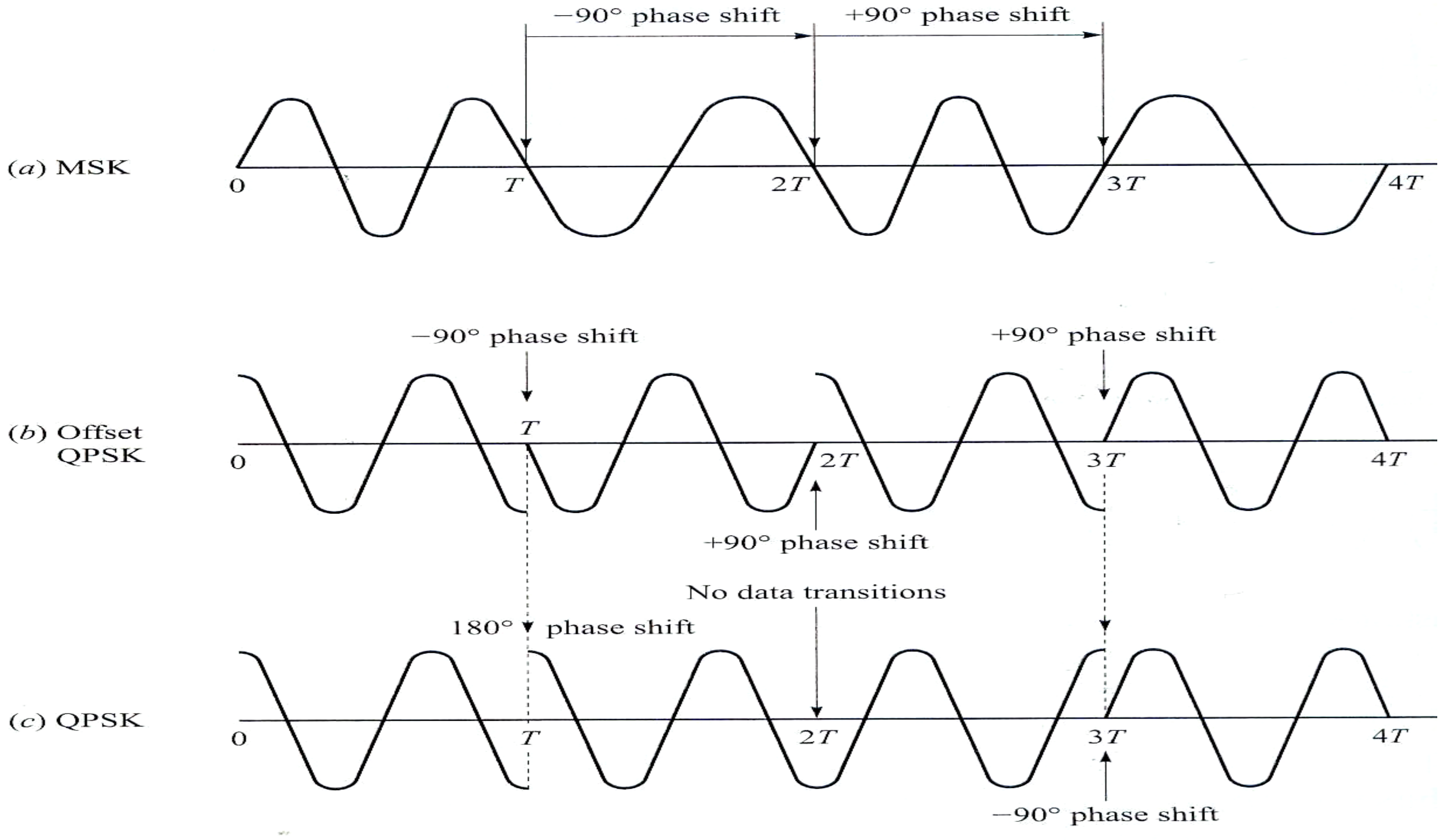
$$s_i(t) = A\cos\left[2\pi f_i t + \theta_n + \frac{1}{2} n\pi(-1)^{i-1}\right], \qquad i = 1, 2$$

# Non-linear Modulation Methods with Memory

- Minimum-shift keying (MSK) (cont.)
  - Why binary CPFSK with $h$=1/2 is called minimum-shift keying (MSK)?
  - Because the frequency separation $\Delta f = f_2 - f_1 = 1/2T$, and $\Delta f = 1/2T$ is the minimum frequency separation that is necessary to ensure the orthogonality of the signals $s_1(t)$ and $s_2(t)$ over a signaling interval of length $T$.

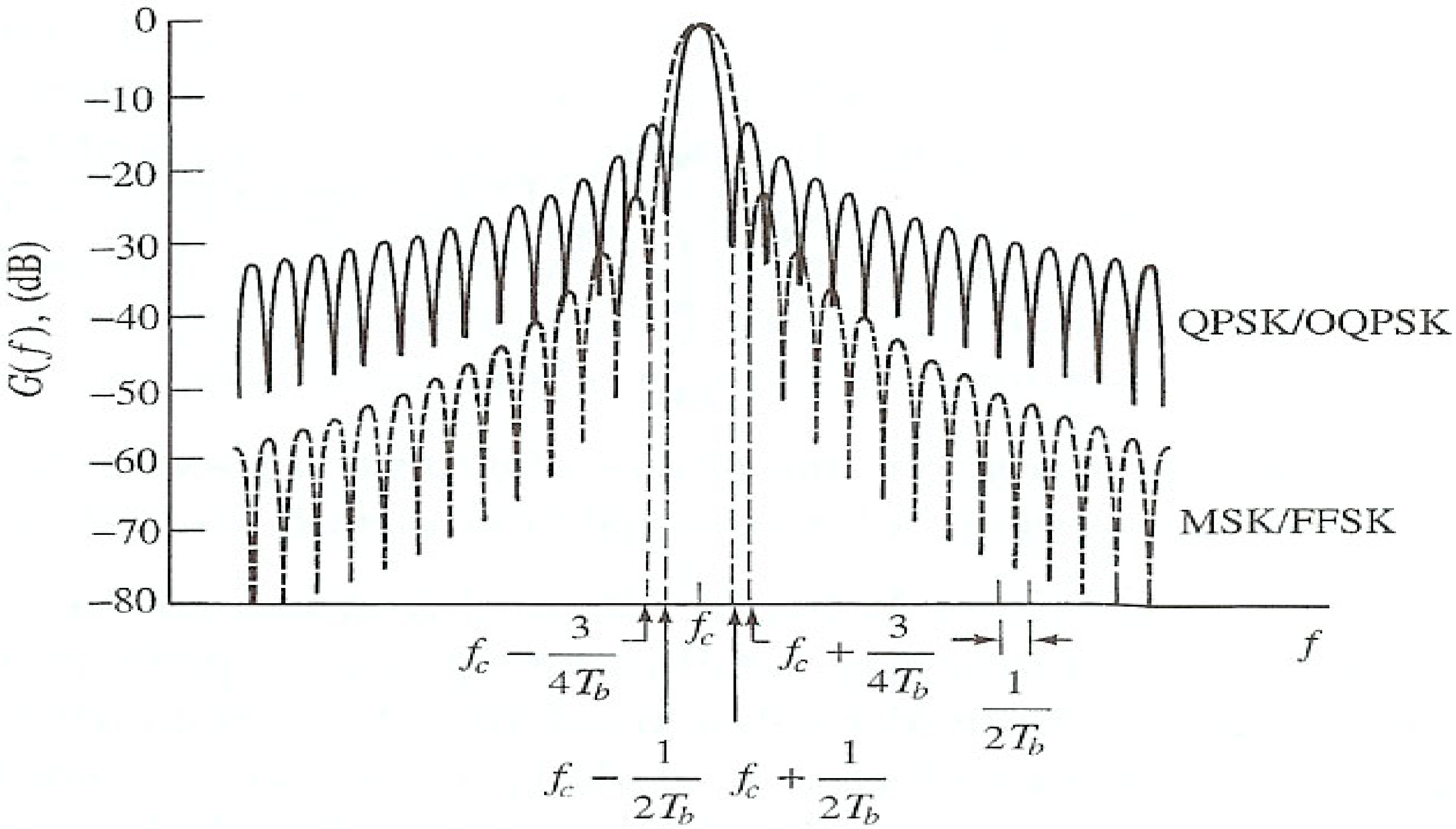
# Non-linear Modulation Methods with Memory

- Minimum-shift keying (MSK) (cont.)
  - Compare the waveforms for MSK with OQPSK and QPSK (cont.)



# Non-linear Modulation Methods with Memory

- Minimum-shift keying (MSK) (cont.)



# Spectral Characteristics of Digitally Modulated Signals

- In most digital communication systems, the available channel bandwidth is limited.
- The system designer must consider the constraints imposed by the channel bandwidth limitation in the selection of the modulation technique used to transmit the information.
- From the power density spectrum, we can determine the channel bandwidth required to transmit the information-bearing signal.

# Power Spectra of Linearly Modulated Signals

- Beginning with the form

$$s(t) = \mathrm{Re}\left[v(t)e^{j2\pi f_c t}\right]$$

where $v(t)$ is the equivalent low-pass signal.

- Autocorrelation function

$$\phi_{ss}(\tau) = \mathrm{Re}\left[\phi_{\upsilon\upsilon}(\tau)e^{j2\pi f_c \tau}\right]$$

- Power density spectrum

$$\Phi_{ss}(f) = \frac{1}{2}[\Phi_{\upsilon\upsilon}(f - f_c) + \Phi_{\upsilon\upsilon}(-f - f_c)]$$

- First we consider the general form

Random Variable

$$\upsilon(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$$

Deterministic

where the transmission rate is $1/T = R/k$ symbols/s and $\{I_n\}$ represents the sequence of symbols.

# Power Spectra of Linearly Modulated Signals

- Autocorrelation function

$$\phi_{\upsilon\upsilon}(t+\tau;t)=\frac{1}{2}E\left[\upsilon^*(t)\upsilon(t+\tau)\right]$$

$$=\frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}E\left[I_n^* I_m\right]g^*(t-nT)g(t+\tau-mT)$$

- We assume the $\{I_n\}$ is WSS with mean $\mu_i$ and the autocorrelation function

$$\phi_{ii}(m)=\tfrac{1}{2}E\left[I_n^* I_{n+m}\right]$$

$$\phi_{\upsilon\upsilon}(t+\tau;t)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\phi_{ii}(m-n)g^*(t-nT)g(t+\tau-mT) \quad \text{let } m'=m-n$$

$$=\sum_{m'=-\infty}^{\infty}\phi_{ii}(m')\sum_{n=-\infty}^{\infty}g^*(t-nT)g\left(t+\tau-\left(m'+n\right)T\right) \quad \text{let } m=m'$$

$$=\sum_{m=-\infty}^{\infty}\phi_{ii}(m)\sum_{n=-\infty}^{\infty}g^*(t-nT)g(t+\tau-nT-mT)$$

# Power Spectra of Linearly Modulated Signals

- The second summation

$$\sum_{n=-\infty}^{\infty} g^*(t-nT)g(t+\tau-nT-mT)$$

is periodic in the *t* variable with period *T*.

- Consequently, $\varphi_{vv}(t+\tau;t)$ is also periodic in the *t* variable with period *T*. That is

$$\phi_{\upsilon\upsilon}(t+T+\tau;t+T)=\phi_{\upsilon\upsilon}(t+\tau;t)$$

- In addition, the mean value of $v(t)$, which is

$$E[\upsilon(t)]=E\left[\sum_{n=-\infty}^{\infty} I_n g(t-nT)\right]=\mu_i \sum_{n=-\infty}^{\infty} g(t-nT)$$

is periodic with period *T*.

# Power Spectra of Linearly Modulated Signals

- Therefore *v*(*t*) is a stochastic process having a periodic mean and autocorrelation function. Such a process is called a *cyclostationary process* or a *periodically stationary process in the wide sense*.
- In order to compute the power density spectrum of a cyclo-stationary process, the dependence of $\varphi_{vv}(t+\tau;t)$ on the *t* variable must be eliminated. Thus,

$$\bar{\phi}_{\upsilon\upsilon}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \phi_{\upsilon\upsilon}(t+\tau;t)dt$$

$$= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} g^*(t-nT)g(t+\tau-nT-mT)dt$$

$$= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2\ -nT}^{T/2\ -nT} g^*(t')g(t'+\tau-mT)dt' \ \left(t'=t-nT\right)$$

# Power Spectra of Linearly Modulated Signals

- We interpret the integral as the *time-autocorrelation function* of $g(t)$ and define it as

$$\phi_{gg}(\tau) = \int_{-\infty}^{\infty} g^*(t) g(t+\tau) dt$$

- Consequently,

$$\overline{\phi}_{\upsilon\upsilon}(\tau) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \overbrace{\phi_{ii}(m)\phi_{gg}(\tau - mT)}^{\text{convolution}}$$

- The (average) power density spectrum of $v(t)$ is in the form

$$\boxed{\Phi_{\upsilon\upsilon}(f) = \frac{1}{T} \left| G(f) \right|^2 \Phi_{ii}(f)}$$

where $G(f)$ is the Fourier transform of $g(t)$, and $\Phi_{ii}(f)$ denotes the power density spectrum of the information sequence

$$\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) e^{-j2\pi f m T}$$

# Power Spectra of Linearly Modulated Signals

- The result illustrates the dependence of the power density spectrum of $v(t)$ on the spectral characteristics of the pulse $g(t)$ and the information sequence $\{I_n\}$.

- That is, the spectral characteristics of $v(t)$ can be controlled by (1) design of the pulse shape $g(t)$ and by (2) design of the correlation characteristics of the information sequence.

- Whereas the dependence of $\Phi_{vv}(f)$ on $G(f)$ is easily understood upon observation of equation, the effect of the correlation properties of the information sequence is more subtle.

- First of all, we note that for an arbitrary autocorrelation $\varphi_{ii}(m)$ the corresponding power density spectrum $\Phi_{ii}(f)$ is periodic in frequency with period $1/T$. (see next page)

# Power Spectra of Linearly Modulated Signals

- In fact, the expression relating the spectrum $\Phi_{ii}(f)$ to the autocorrelation $\varphi_{ii}(m)$ is in the form of an exponential Fourier series with the $\{\varphi_{ii}(m)\}$ as the Fourier coefficients.

$$\phi_{ii}(m) = T\int_{-1/2T}^{1/2T} \Phi_{ii}(f) e^{j2\pi fmT} df$$

- Second, let us consider the case in which the information symbols in the sequence are real and mutually uncorrelated. In this case, the autocorrelation function $\varphi_{ii}(m)$ can be expressed as (applying Chapter 2, page 96, $\mu(t_1,t_2)=\mu(t_1-t_2)=\mu(\tau)=\phi(\tau)-m^2$ )

$$\phi_{ii}(m) = \begin{cases} \sigma_i^2+\mu_i^2 & (m=0) \\ \mu_i^2 & (m\neq 0) \end{cases}$$

where $\sigma_i^2$ denotes the variance of an information symbol.

# Power Spectra of Linearly Modulated Signals

- Substitute for $\varphi_{ii}(m)$ in equation, we obtain

$$\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) e^{-j2\pi fmT}$$

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

where $\omega_s = \dfrac{2\pi}{T_s}$

$$= \sigma_i^2 + \mu_i^2 \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT}$$

$$= \sigma_i^2 + \frac{\mu_i^2}{T} \sum_{m=-\infty}^{\infty} \delta(f - \frac{m}{T})$$

It may be viewed as the exponential Fourier series of a periodic train of impulses with each impulse having an area $1/T$.

- The desired result for the power density spectrum of $v(t)$ when the sequence of information symbols is uncorrelated.

$$\Phi_{\upsilon\upsilon}(f) = \frac{\sigma_i^2}{T} \left| G(f) \right|^2 + \frac{\mu_i^2}{T^2} \sum_{m=-\infty}^{\infty} \left| G\left(\frac{m}{T}\right) \right|^2 \delta\left(f - \frac{m}{T}\right)$$

# Power Spectra of Linearly Modulated Signals

- The expression for the power density spectrum is purposely separated into two terms to emphasize the two different types of spectral components.

- The first term is the continuous spectrum, and its shape depends only on the spectral characteristic of the signal pulse $g(t)$.

- The second term consists of discrete frequency components spaced $1/T$ apart in frequency. Each spectral line has a power that is proportional to $|G(f)|^2$ evaluated at $f = m/T$.

- Note that the discrete frequency components vanish when the information symbols have zero mean, i.e., $\mu_i=0$. This condition is usually desirable for the digital modulation techniques under consideration, and it is satisfied when the information symbols are equally likely and symmetrically positioned in the complex plane.

# Power Spectra of Linearly Modulated Signals

- Example To illustrate the spectral shaping resulting from $g(t)$, consider the rectangular pulse shown in figure. The Fourier transform of $g(t)$ is

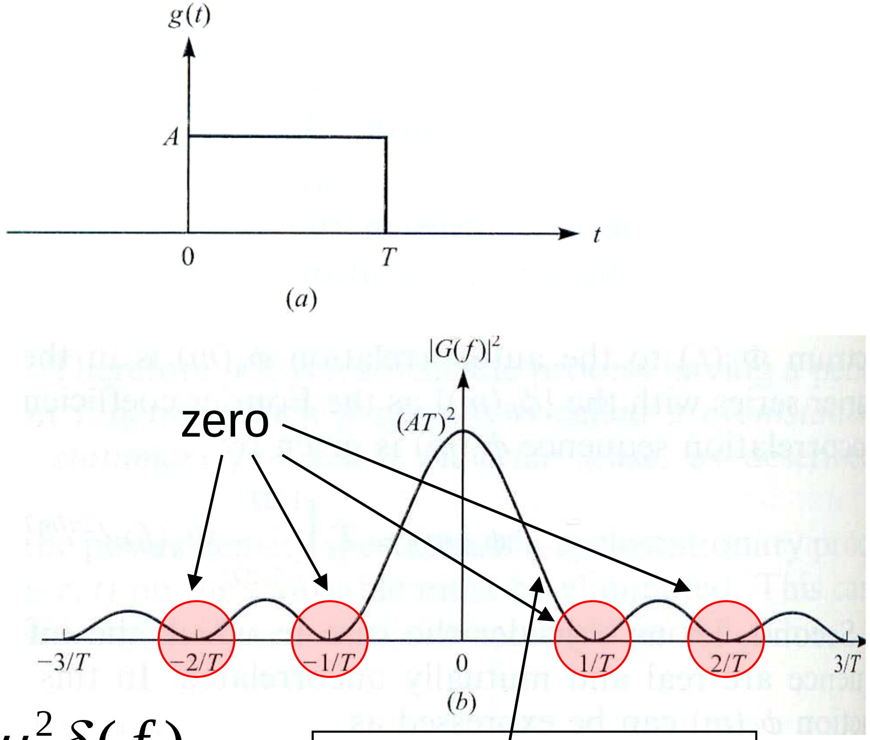
$$G(f) = AT\frac{\sin \pi fT}{\pi fT}e^{-j\pi fT}$$

Hence

$$\left|G(f)\right|^2 = (AT)^2\left(\frac{\sin \pi fT}{\pi fT}\right)^2$$

Thus

$$\Phi_{\upsilon\upsilon}(f) = \sigma_i^2 A^2 T\left(\frac{\sin \pi fT}{\pi fT}\right)^2 + A^2\mu_i^2\delta(f)$$



zero

Decays inversely as the square of the frequency

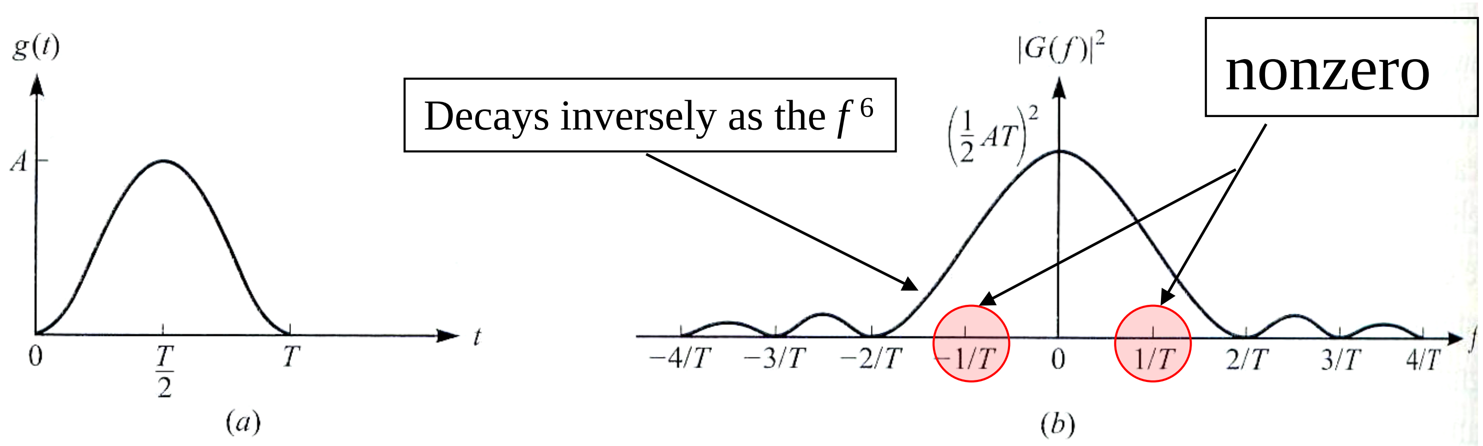
# Power Spectra of Linearly Modulated Signals

- **Example** As a second illustration of the spectral shaping resulting from $g(t)$, we consider the raised cosine pulse

$$g(t)=\frac{A}{2}\left[1+\cos\frac{2\pi}{T}\left(t-\frac{T}{2}\right)\right],\qquad 0\le t\le T$$

its Fourier transform is:

$$G(f)=\frac{A}{2}\frac{\sin\pi fT}{\pi fT(1-f^2T^2)}e^{-j\pi fT}$$



(a)

(b)

Decays inversely as the $f^6$

nonzero

# Power Spectra of Linearly Modulated Signals

- Example To illustrate that spectral shaping can also be accomplished by operations performed on the input information sequence, we consider a binary sequence $\{b_n\}$ from which we form the symbols $I_n = b_n + b_{n-1}$
  - The $\{b_n\}$ are assumed to be uncorrelated random variables, each having zero mean and unit variance. Then the autocorrelation function of the sequence $\{I_n\}$ is

$E[X_iX_j]=E[X_i]E[X_j]$

$E[(X_i\text{-}0)^2]=1$

$$\phi_{ii}(m) = E(I_n I_{n+m}) = E\left[\left(b_n + b_{n-1}\right)\left(b_{n+m} + b_{n+m-1}\right)\right]$$

$$= \begin{cases} E\left[b_n^2 + 2b_n b_{n-1} + b_{n-1}^2\right] & (m=0) \\ E\left[b_n^2 + b_n b_{n+1} + b_{n-1} b_{n+1} + b_n b_{n-1}\right] & (m=+1) \\ E\left[b_n b_{n-1} + b_n b_{n-2} + b_{n-1}^2 + b_{n-1} b_{n-2}\right] & (m=\text{-}1) \\ E\left[b_n b_{n+m} + b_n b_{n+m-1} + b_{n-1} b_{n+m} + b_{n-1} b_{n+m-1}\right] & (\text{otherwise}) \end{cases} = \begin{cases} 2 & (m=0) \\ 1 & (m=\pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

# Power Spectra of Linearly Modulated Signals

- Example (cont.)

Hence, the power density spectrum of the input sequence is

$$\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) e^{-j2\pi fmT}$$
$$= 2(1+\cos 2\pi fT)$$
$$= 4\cos^2 \pi fT$$

and the corresponding power density spectrum for the (low-pass) modulated signal is

$$\Phi_{\upsilon\upsilon}(f) = \frac{4}{T} |G(f)|^2 \cos^2 \pi fT$$