Chapter 4
Phase and Frequency Modulation

Wireless Information Transmission System Lab.
Institute of Communications Engineering
National Sun Yat-sen University

Outline

- 4.1 Introduction
- 4.2 Basic Definitions
- 4.3 Frequency Modulation
- 4.4 Phase-locked Loop

Chapter 4.1 Introduction

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4.1 Introduction

- In this chapter, we study a second family of continuous-wave(CW) modulation systems, namely, *angle modulation*, in which the angle of the carrier wave is varied according to the baseband signals.
- In this method of modulation, the amplitude of the carrier wave is maintained constant.
- There are two common forms of angle modulation, namely, *phase modulation* and *frequency modulation*.
- An important feature of angle modulation is that it can provide better discrimination against noise and interference than amplitude modulation.

4.1 Introduction

- However, this improvement in performance is achieved at the expense of increased transmission bandwidth.

- Moreover, the improvement in the noise performance with angle modulation is achieved at the expense of increased system complexity in both the transmitter and receiver.

- Such a trade-off is not possible with amplitude modulation.

Chapter 4.2
Basic Definitions

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4.2 Basic Definitions

- Let $\theta_i(t)$ denote the angle of a modulated sinusoidal carrier at time t; it is assumed to be a function of the information–bearing signal or message signal.
- We express the resulting angle-modulated wave as

$$s(t) = A_c \cos\left[\theta_i(t)\right] \tag{4.1}$$

where A_c is the carrier amplitude.

- The *average frequency* in Hertz over an interval from t to $t+\Delta t$ is given by

$$f_{\Delta t}(t) = \frac{\theta_i(t+\Delta t) - \theta_i(t)}{2\pi\Delta t} \tag{4.2}$$

- The *instantaneous frequency* of the angle-modulated signal $s(t)$:

$$f_i(t) = \lim_{\Delta t \to 0} f_{\Delta t}(t) = \lim_{\Delta t \to 0}\left[\frac{\theta_i(t+\Delta t) - \theta_i(t)}{2\pi\Delta t}\right] = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$

4.2 Basic Definitions

- For an *unmodulated carrier*, the angle $\theta_i(t)$ is given by

$$\theta_i(t) = 2\pi f_c t + \phi_c$$

and corresponding phasor rotates with a constant angular velocity equal to $2\pi f_c$. The constant ϕ_c is the value of $\theta_i(t)$ at t=0.

- There are an infinite number of ways in which the angle $\theta_i(t)$ may be varied in some manner with the message (baseband) signal.

- We shall consider only two commonly used methods, *phase modulation* and *frequency modulation*.

4.2 Basic Definitions

- **Phase modulation (PM)** is that form of angle modulation in which the instantaneous angle $\theta_i(t)$ is varied linearly with the message signal as shown by

$$\theta_i(t) = 2\pi f_c t + k_p m(t) \tag{4.4}$$

The term $2\pi f_c t$ represents the angle of the unmodulated carrier; k_p represents the *phase sensitivity* of the modulator, expressed in *radians per volt* on the assumption that $m(t)$ is a voltage waveform.

For convenience, we have assumed in Eq. (4.4) that the angle of the unmodulated carrier is zero at t=0. The phase-modulated signal $s(t)$ is thus described in the time domain by

$$\boxed{s(t) = A_c \cos\left[2\pi f_c t + k_p m(t)\right]} \tag{4.5}$$

4.2 Basic Definitions

- **Frequency modulation (FM)** is that form of angle modulation in which the instantaneous frequency $f_i(t)$ is varied linearly with the message signal $m(t)$, as shown by

$$f_i(t) = f_c + k_f m(t) \tag{4.6}$$

f_c : The frequency of the unmodulated carrier

k_f : The *frequency sensitivity* of the modulator (Hertz per volt)

Integrating Eq. (4.6) with respect to time and multiplying the result by 2π, we get

$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau \tag{4.7}$$

where, for convenience, we have assumed that the angle of the unmodulated carrier wave is zero at t=0. The frequency-modulated signal is therefore described in the time domain by

$$\boxed{s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]} \tag{4.8}$$

4.2 Basic Definitions

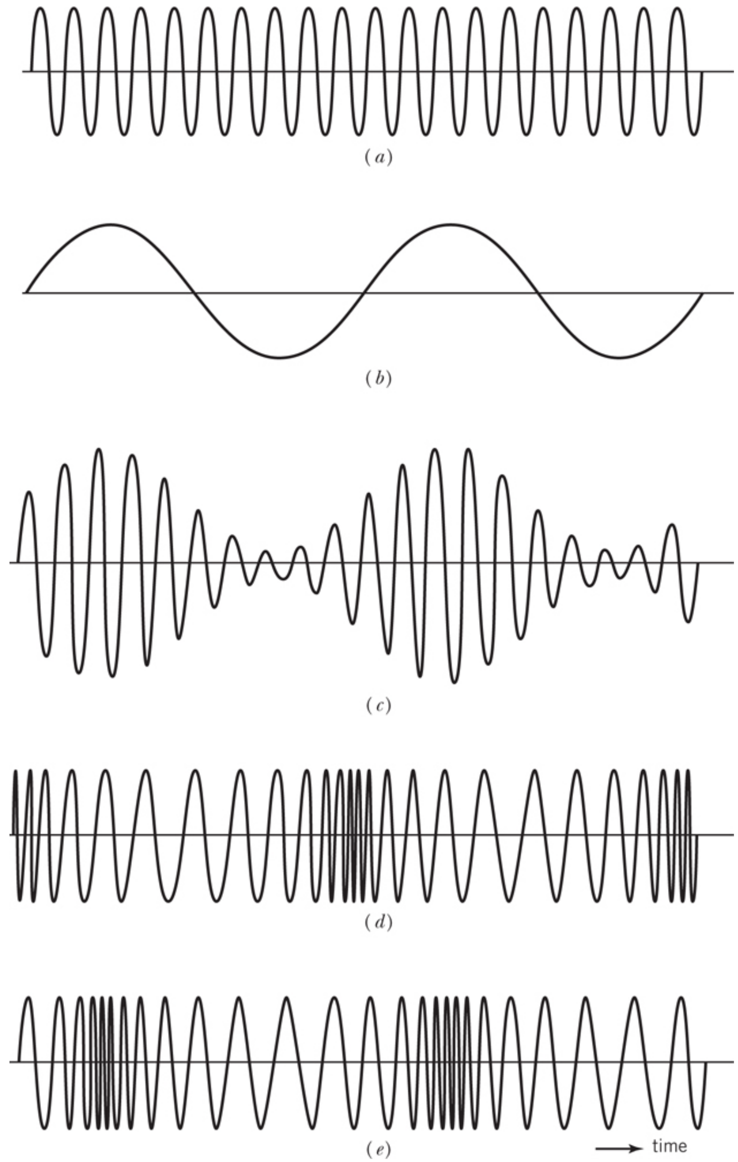
a) Carrier wave

b) Sinusoidal modulating signal

c) Amplitude-modulated signal

d) Phase-modulated signal

e) Frequency-modulated signal



Properties of Angle-Modulated Waves

- Property 1: Constancy of Transmitted Power:
  - From both Eqs. (4.4) and (4.7), we readily see that the amplitude of PM and FM waves is maintained at a constant value equal to the carrier amplitude A_c for all time t, irrespective of the sensitivity factors k_p and k_f .
  - Consequently, the average transmitted power of angle-modulated waves is a constant, as shown by

$$P_{av} = \frac{1}{2} A_c^2 \tag{4.9}$$

where it is assumed that the load resistance is 1 ohm.

$$\boxed{\left(P = \frac{V^2}{R} \right)}$$

Properties of Angle-Modulated Waves

- Property 2: Nonlinearity of the Modulation Process
  - Both PM and FM waves violate the principle of superposition.
  - For example, the message signal $m(t)$ is made up of two different components, $m_1(t)$ and $m_2(t)$: $m(t) = m_1(t) + m_2(t)$
  - Let $s(t)$, $s_1(t)$, and $s_2(t)$ denote the PM waves produced by $m(t)$, $m_1(t)$, and $m_2(t)$ in accordance with Eq. (4.4), respectively. We may express these PM waves as follows:

$$\boxed{\theta_i(t) = 2\pi f_c t + k_p m(t) \quad (4.4)}$$

$$s(t) = A_c \cos\left[2\pi f_c t + k_p\left(m_1(t) + m_2(t)\right)\right]$$

$$s_1(t) = A_c \cos\left[2\pi f_c t + k_p m_1(t)\right]$$

$$s_2(t) = A_c \cos\left[2\pi f_c t + k_p m_2(t)\right]$$

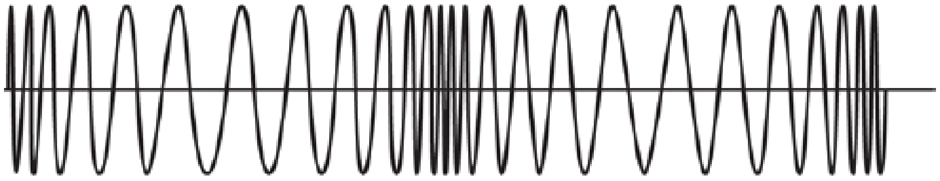
$$\Rightarrow \quad m(t) = m_1(t) + m_2(t)$$

$$s(t) \neq s_1(t) + s_2(t)$$

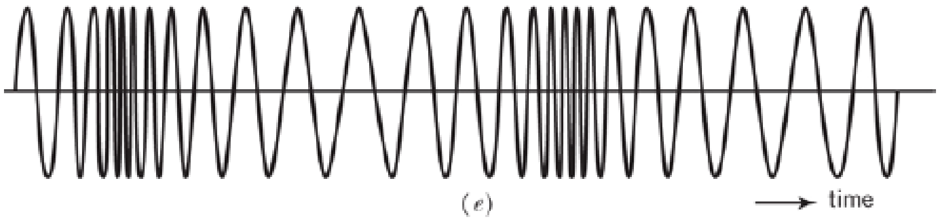
  - Frequency modulation offers superior noise performance compare to amplitude modulation,

Properties of Angle-Modulated Waves

- Property 3: Irregularity of Zero-Crossings
  - *Zero-crossing* are defined as the instants of time at which a waveform changes its amplitude from positive to negative value or the other way around.
  - The zero-crossings of a PM or FM wave no longer have a perfect regularity in their spacing across the time-scale.
  - The irregularity of zero-crossings in angle-modulated waves is attributed to the nonlinear character of the modulation process.



(d)



(e)

Properties of Angle-Modulated Waves

- Property 4: Visualization Difficulty of Message Waveform
  - In AM, we see the message waveform as the envelope of the modulated wave, provided the percentage modulation is less than 100 percent.

    (AM: The percentage modulation over 100 percent→phase reversal→distortion)

  - This is not so in angle modulation, as illustrated by the corresponding waveform of Figures 4.1*d* and 4.1*e* for PM and FM, respectively.

Properties of Angle-Modulated Waves

- Property 5-Trade-OFF of Increased Transmission Bandwidth for Improved Noise Performance
  - An important advantage of angle modulation over amplitude modulation is the realization of improved noise performance.
  - This advantage is attributed to the fact that the transmission of a message signal by modulating the angle of a sinusoidal carrier wave is less sensitive to the presence of additive noise than transmission by modulating the amplitude of the carrier.
  - The improvement in noise performance is achieved at the expense of a corresponding increase in the transmission bandwidth requirement of angle modulation.

Properties of Angle-Modulated Waves

- Property 5-Trade-OFF of Increased Transmission Bandwidth for Improved Noise Performance
  - The use of angle modulation offers the possibility of exchanging an increase in the transmission bandwidth for an improvement in noise performance.
  - Such a trade-off is not possible with amplitude modulation since the transmission bandwidth of an amplitude-modulated wave is fixed somewhere between the message bandwidth W and $2W$, depending on the type of modulation employed.

Example 4.1 Zero-Crossings

- Consider a modulating wave $m(t)$ that increases linearly with time t, starting at t=0, as shown by

$$m(t)=\begin{cases} at, & t\geq 0 \\ 0, & t<0 \end{cases}$$

where a is the slope parameter (see Figure 4.2a). In what follows, we study the zero-crossings of the PM and FM waves produced by $m(t)$ for the following set of parameters:

$$f_c=\frac{1}{4}\text{Hz}$$

$$a=1 \text{ volt/s}$$

Example 4.1 Zero-Crossings

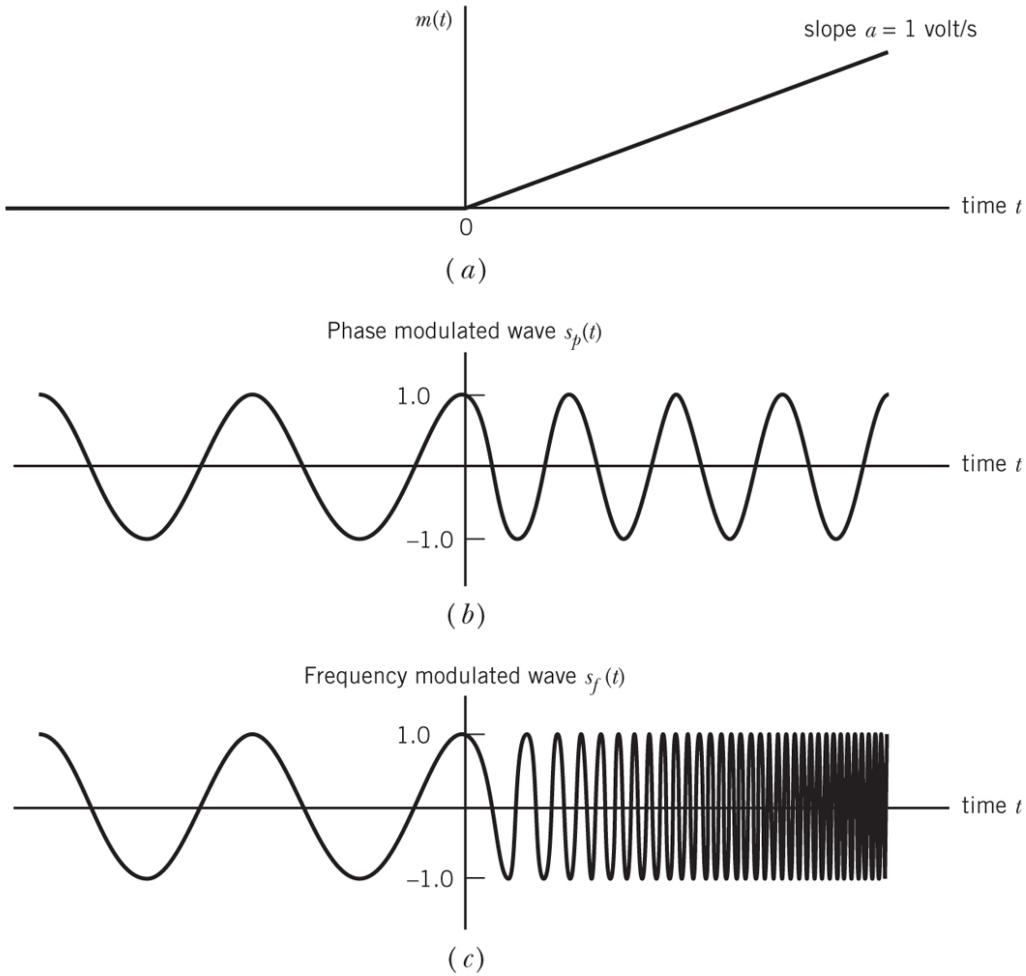


Fig. 4.2 Starting at time $t = 0$, the figure displays (a) linearly increasing message signal $m(t)$, (b)phase-modulated wave, and (c) frequency-modulated wave.

Example 4.1 Zero-Crossings

- Phase Modulation:
  - Phase-sensitivity factor $k_p = \pi/2$ radians/volt. Applying Eq. (4.5) to the given $m(t)$ yields the PM wave

$$\boxed{s(t) = A_c \cos\left[2\pi f_c t + k_p m(t)\right] \quad (4.5)}$$

$$s(t) = \begin{cases} A_c \cos\left(2\pi f_c t + k_p a t\right), & t \ge 0 \\ A_c \cos\left(2\pi f_c t\right), & t < 0 \end{cases}$$

  which is plotted in Figure 4.2*b* for A_c=1 volt.

  - Let t_n denote the instant of time at which the PM wave experiences a zero crossing; this occurs whenever the angle of the PM wave is an odd multiple of $\pi/2$:

$$2\pi f_c t_n + k_p a t_n = \pi\left(2 f_c + \frac{k_p a}{\pi}\right) t_n = \frac{\pi}{2} + n\pi, \quad n = 0, 1, 2, \ldots$$

$$t_n = \frac{\frac{1}{2} + n}{2 f_c + \frac{k_p}{\pi} a} \Rightarrow t_n = \frac{1}{2} + n, \quad n = 0, 1, 2, \ldots$$

Example 4.1 Zero-Crossings

- Frequency Modulation:
  - Frequency-sensitivity factor, k_f=1 Hz/volt. Applying Eq. (4.8) yields the FM wave

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau\right] \quad (4.8)$$

$$s(t) = \begin{cases} A_c \cos\left(2\pi f_c t + \pi k_f a t^2\right), & t \geq 0 \\ A_c \cos\left(2\pi f_c t\right), & t < 0 \end{cases}$$

  which is plotted in Figure 4.2*c*.

  - Invoking the definition of a zero-crossing, we can obtain:

$$2\pi f_c t_n + \pi k_f a t_n^2 = \frac{\pi}{2} + n\pi, \quad n = 0,1,2,\ldots$$

$$t_n = \frac{1}{a k_f}\left(-f_c + \sqrt{f_c^2 + a k_f\left(\frac{1}{2} + n\right)}\right), \quad n = 0,1,2,\ldots$$

$$t_n = \frac{1}{4}\left(-1 + \sqrt{9 + 16n}\right), \quad n = 0,1,2,\ldots$$

Example 4.1 Zero-Crossings

- Comparing the zero-crossing results derived for PM and FM waves, we may make the following observations once the linear modulating wave begins to act on the sinusoidal carrier wave:

1. For PM, regularity of the zero-crossings is maintained; the instantaneous frequency changes from the unmodulated value of f_c=1/4 Hz to the new constant value of $f_c + k_p\left(a/2\pi\right) = 0.5\text{Hz}$

2. For FM, the zero-crossings assume an irregular form; as expected, the instantaneous frequency increases linearly with time t.

4.2 Basic Definitions

- Comparing Eq. (4.5) with (4.8) reveals that an FM signal may be regarded as a PM signal in which the modulating wave is $\int_0^t m(\tau)\,d\tau$ in place of *m*(*t*).

$$s(t) = A_c \cos\left[2\pi f_c t + k_p m(t)\right] \quad (4.5)$$

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau\right] \quad (4.8)$$

- The FM signal can be generated by first integrating *m*(*t*) and then using the result as the input to a phase modulator, as in Figure 4.3*a*.
- Conversely, a PM signal can be generated by first differentiating *m*(*t*) and then using the result as the input to a frequency modulator, as in Figure 4.3*b*.
- We may thus deduce all the properties of PM signals from those of FM signals and vice versa. Henceforth, we concentrate our attention on FM signals.

4.2 Basic Definitions

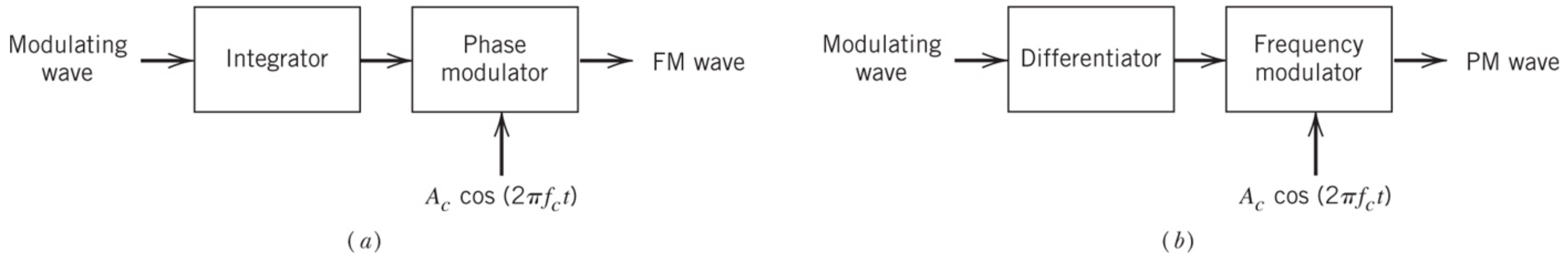


Figure 4.3 Illustrating the relationship between frequency modulation and phase modulation. (a) Scheme for generating an FM wave by using a phase modulator, (b) scheme for generating a PM wave by using a frequency modulator.

	$\theta_i(t)$	$f_i(t)$
Unmodulated signal	$2\pi f_c t$	f_c
PM signal	$2\pi f_c t + k_p m(t)$	$f_c + \frac{k_p}{2\pi}\frac{dm(t)}{dt}$
FM signal	$2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau$	$f_c + k_f m(t)$

Chapter 4.3
Frequency Modulation

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4.3 Frequency Modulation

- The FM signal $s(t)$ define by Eq. (4.8) is a nonlinear function of the modulating signal $m(t)$, which makes frequency modulation a nonlinear modulation process.

$$\boxed{s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau\right] \quad (4.8)}$$

- How then can we tackle the spectral analysis of FM signal? We propose to provide an empirical answer to this important question by proceeding in the same manner as with AM modulation, that is, we consider the simplest case possible, namely, single-tone modulation.

- Consider then a sinusoidal modulating signal define by

$$m(t) = A_m \cos(2\pi f_m t) \qquad (4.10)$$

4.3 Frequency Modulation

- The instantaneous frequency of the resulting FM signal is

$$f_i(t) = f_c + k_f A_m \cos(2\pi f_m t) = f_c + \Delta f \cos(2\pi f_m t) \qquad (4.11)$$

$$\Delta f = k_f A_m \qquad (4.12)$$

  - The quantity Δf is called the *frequency deviation*, representing the maximum departure of the instantaneous frequency of the FM signal from the carrier frequency f_c.
  - A fundamental characteristic of an FM signal is that the frequency deviation Δf is proportional to the amplitude of the modulating signal and is independent of the modulating frequency.

- Using Eq. (4.11), the angle $\theta_i(t)$ of the FM signal is obtained as

$$\theta_i(t) = 2\pi \int_0^t f_i(t)\,dt = 2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t)$$

- The ratio of the frequency deviation Δf to the modulation frequency f_m is commonly called the *modulation index* of the FM signal.

4.3 Frequency Modulation

- The modulation index is denoted by β: $\beta = \dfrac{\Delta f}{f_m}$

$$\theta_i(t) = 2\pi f_c t + \beta \sin(2\pi f_m t)$$

- The parameter β represents the phase deviation of the FM signal, i.e. the maximum departure of the angle $\theta_i(t)$ from the angle $2\pi f_c t$ of the unmodulated carrier. β is measured in radians.
- The FM signal itself is given by

$$s(t) = A_c \cos\left[2\pi f_c t + \beta \sin(2\pi f_m t)\right] \tag{4.16}$$

Depending on the value of the modulation index β, we may distinguish two cases of frequency modulation:

  - *Narrow-band FM*, for which β is small compared to one radian.
  - *Wide-band FM*, for which β is large compared to one radian.

4.3 Frequency Modulation

- Narrow-band frequency modulation
  - Consider Eq. (4.16), which defines an FM signals resulting form the use of sinusoidal modulating signal. Expanding this relation, we get

$$s(t) = A_c \cos(2\pi f_c t)\cos\left[\beta \sin(2\pi f_m t)\right] - A_c \sin(2\pi f_c t)\sin\left[\beta \sin(2\pi f_m t)\right] \quad (4.17)$$

  - Assuming that the modulation index β is small compared to one radian, we may use the following two approximations:

$$\cos\left[\beta \sin(2\pi f_m t)\right] \simeq 1 \qquad \sin\left[\beta \sin(2\pi f_m t)\right] \simeq \beta \sin(2\pi f_m t)$$

$$s(t) \simeq A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\sin(2\pi f_m t) \quad (4.18)$$

$$s(t) \simeq A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c \left\{\cos\left[2\pi(f_c + f_m)t\right] - \cos\left[2\pi(f_c - f_m)t\right]\right\} \quad (4.19)$$

$$\boxed{\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]}$$

4.3 Frequency Modulation

- This expression is somewhat similar to the corresponding one defining an AM signal (from Example 3.1):

$$s_{AM}(t) = A_c \cos(2\pi f_c t) + \frac{1}{2}\mu A_c \left\{\cos\left[2\pi(f_c + f_m)t\right] + \cos\left[2\pi(f_c - f_m)t\right]\right\} \quad (4.20)$$

where μ is the modulation factor of the AM signal.

- Compare Eqs. (4.19) and (4.20), we see that the basic difference between an AM signal and a narrow-band FM signal is that the algebraic sign of the lower side frequency in the narrow-band FM is reversed.

- <u>Thus, a narrow-band FM signal requires essentially the same transmission bandwidth (i.e. $2f_m$) as the AM signal.</u>

4.3 Frequency Modulation

- Wide-band frequency modulation
  - The following studies the spectrum of the single-tone FM signal of Eq. (4.16) for an arbitrary value of the modulation index β.

$$s(t) = A_c \cos\left[2\pi f_c t + \beta \sin\left(2\pi f_m t\right)\right] \qquad (4.16)$$

  - By using the complex representation of band-pass signals described in Chapter 2: (Carrier frequency f_c compared to the bandwidth of the FM signal is large enough)

$$s(t) = \mathrm{Re}\left[A_c \exp\left(j2\pi f_c t + j\beta \sin\left(2\pi f_m t\right)\right)\right] \qquad (4.21)$$

$$= \mathrm{Re}\left[\tilde{s}(t)\exp\left(j2\pi f_c t\right)\right]$$

where $\tilde{s}(t) = A_c \exp\left[j\beta \sin\left(2\pi f_m t\right)\right] \rightarrow$ periodic function of t

4.3 Frequency Modulation

- Wide-band frequency modulation
  - We may therefore expend $\tilde{s}(t)$ in the form of complex Fourier series as follows:

$$\tilde{s}(t)=\sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_m t) \tag{4.23}$$

$$c_n = f_m \int_{-1/2f_m}^{1/2f_m} \tilde{s}(t)\exp(-j2\pi n f_m t)\,dt$$

$$= f_m A_c \int_{-1/2f_m}^{1/2f_m} \exp\left[j\beta\sin(2\pi f_m t) - j2\pi n f_m t\right]dt \tag{4.24}$$

$x = 2\pi f_m t$

$$c_n = \frac{A_c}{2\pi}\int_{-\pi}^{\pi}\exp\left[j(\beta\sin x - nx)\right]dx \tag{4.26}$$

$$c_n = A_c J_n(\beta) \quad \because J_n(\beta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[j(\beta\sin x - nx)\right]dx \tag{4.28}$$

*n*th order Bessel function of the first kind.

$$s(t) = A_c \cdot \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} J_n(\beta)\exp\left[j2\pi(f_c + n f_m)t\right]\right] \tag{4.31}$$

4.3 Frequency Modulation

- Taking the Fourier transforms of both sides of Eq. (4.31)

$$S(f)=\frac{A_c}{2}\sum_{n=-\infty}^{\infty}J_n(\beta)\left[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)\right] \qquad (4.32)$$

- In Figure 4.6 we have plotted the Bessel function $J_n(\beta)$ versus the modulation index β for different positive integer values of n.

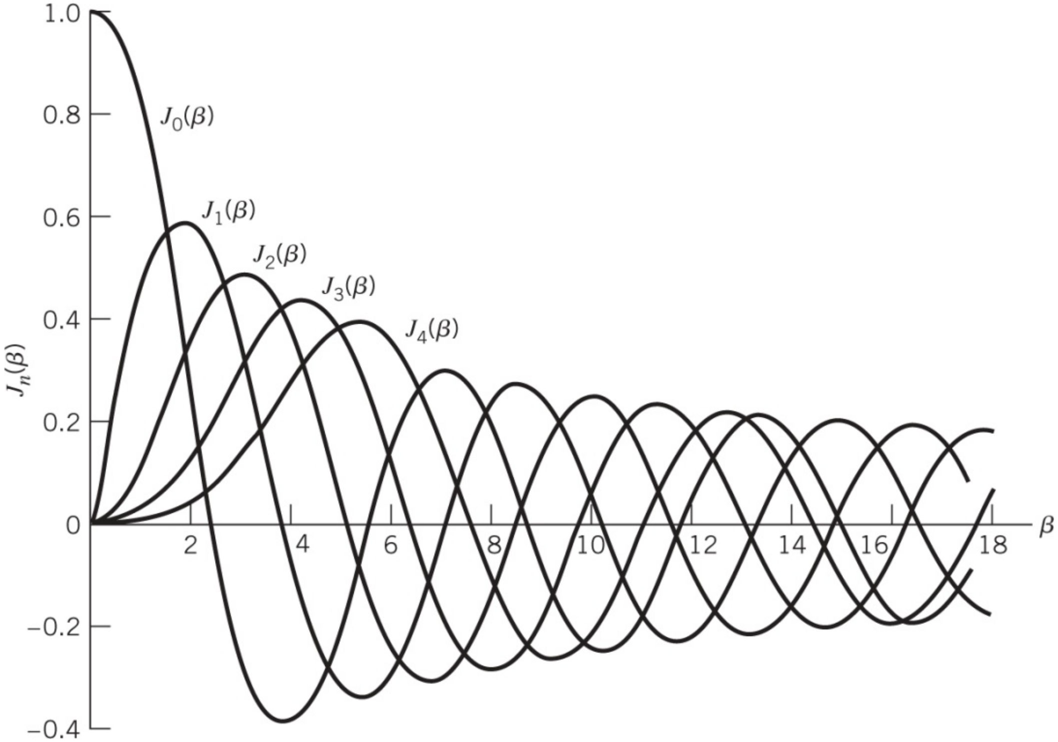


FIGURE4.6 Plots of Bessel functions of the first kind.

4.3 Frequency Modulation

- We can develop further insight into the behavior of the Bessel function $J_n(\beta)$ by making use of the following properties:

1. For n even, we have $J_n(\beta)=J_{-n}(\beta)$; on the other hand, for n odd, we have $J_n(\beta)=-J_{-n}(\beta)$. That is

$$J_n(\beta)=(-1)^n J_{-n}(\beta) \quad \text{for all } n \tag{4.33}$$

2. For small values of the modulation index β, we have

$$\left.\begin{aligned} J_0(\beta) &\simeq 1 \\ J_1(\beta) &\simeq \frac{\beta}{2} \\ J_n(\beta) &\simeq 0, \qquad n>2 \end{aligned}\right\} \tag{4.34}$$

3.

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta)=1 \tag{4.35}$$

4.3 Frequency Modulation

- Thus, using Eqs. (4.32) through (4.35) and the curves of Figure 4.6, we may make the following observations:

1. The spectrum of an FM signal contains a carrier component (n=0) and an infinite set of side frequencies located symmetrically on either side of the carrier at frequency separations of f_m, $2f_m$, $3f_m$, ….

   (An AM system gives rise to only one pair of side frequencies.)

2. For the special case of β small compared with unity, only the Bessel coefficients $J_0(\beta)$ and $J_1(\beta)$ have significant values (see 4.34), so that the FM signal is effectively composed of a carrier and a single pair of side frequencies at $f_c \pm f_m$.

   (This situation corresponds to the special case of narrowband FM that was considered previously)

4.3 Frequency Modulation

3. The amplitude of the carrier component of an FM signal is dependent on the modulation index β. The physical explanation for this property is that the envelope of an FM signal is constant, so that the average power of such a signal developed across a 1–ohm resistor is also constant, as shown by

$$P = \frac{1}{2} A_c^2 \quad \text{(Using (4.31) and (4.35))} \tag{4.36}$$

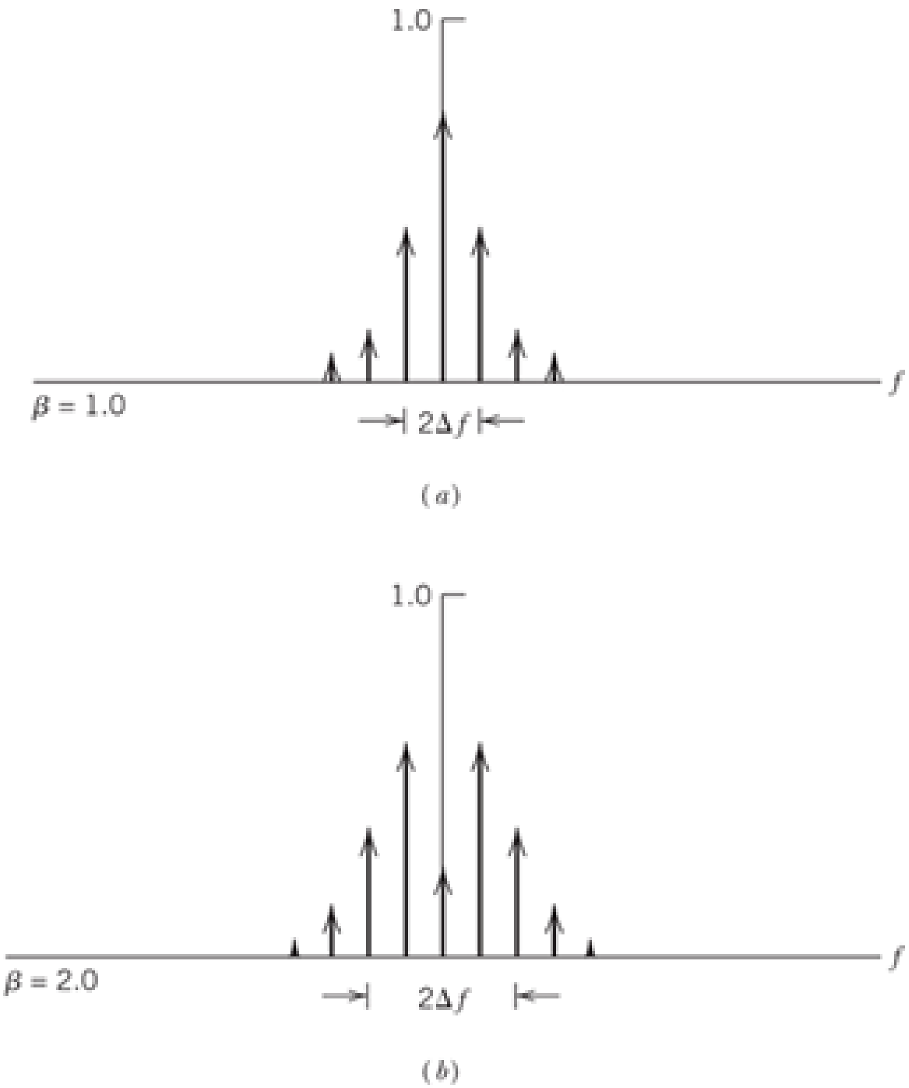
EXAMPLE 4.3 Spectra of FM Signals

- In this example, we wish to investigate the ways in which variations in the amplitude and frequency of a sinusoidal modulating signal affect the spectrum of the FM signal.

- Consider first the case when the frequency of the modulating signal is fixed, but its amplitude is varied, producing a corresponding variation in the frequency deviation Δf.

- Consider next the case when the amplitude of the modulating signal is fixed; that is, the frequency deviation Δf is maintained constant, and the modulation frequency f_m is varied.

EXAMPLE 4.3 Spectra of FM Signals



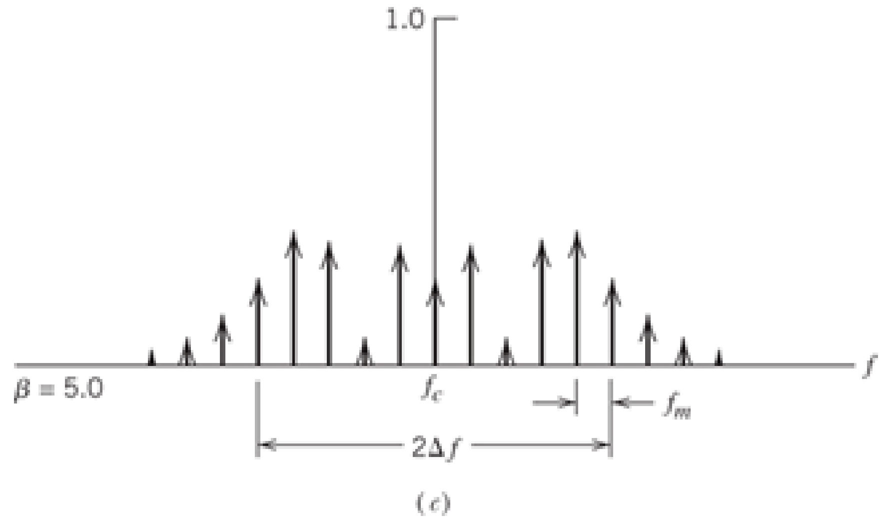
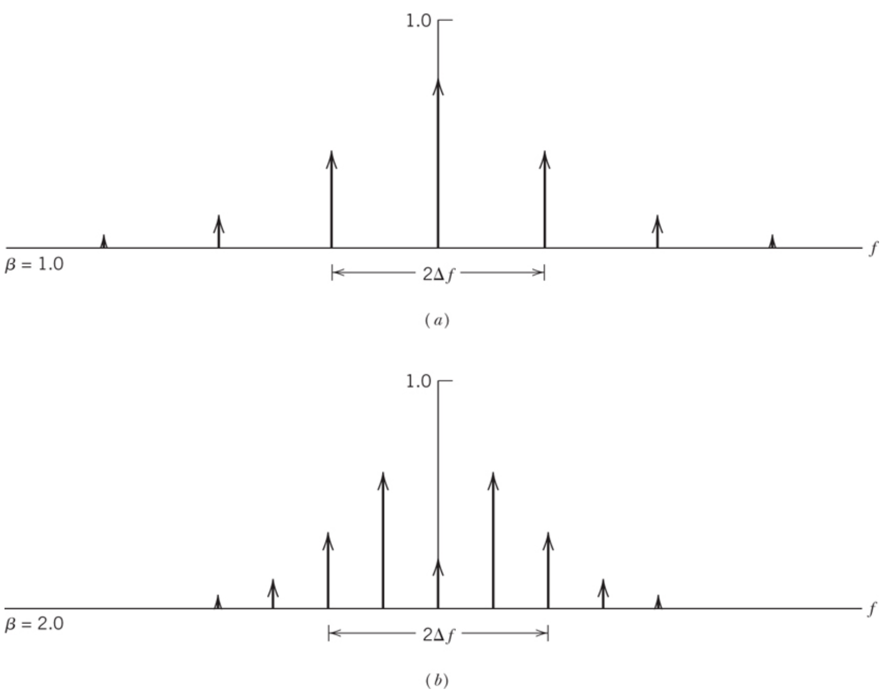


FIGURE4.7 Discrete amplitude spectra of an FM signal, normalized with respect to the carrier amplitude, for the case of sinusoidal modulation of fixed frequency and varying amplitude. Only the spectra for positive frequencies are shown.

EXAMPLE 4.3 Spectra of FM Signals

- We have an increasing number of spectral lines crowding into the fixed frequency interval $f_c - \Delta f < |f| < f_c + \Delta f$.
- When β approaches infinity, the bandwidth of the FM wave approaches the limiting value of $2\Delta f$, which is an important point to keep in mind.



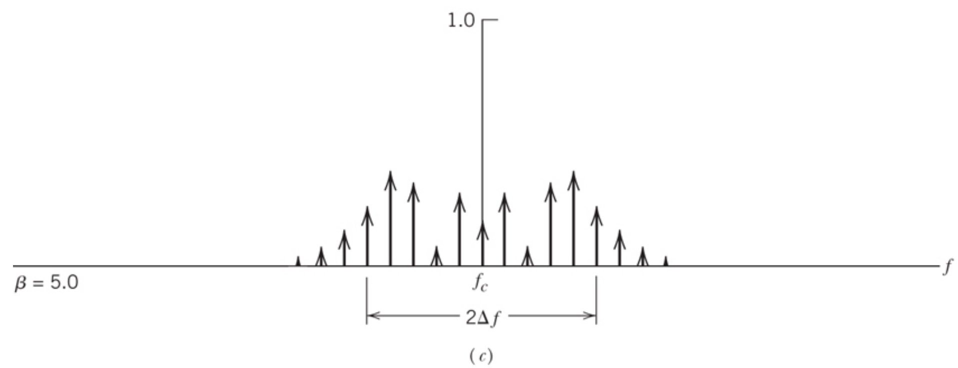


FIGURE 4.8 Discrete amplitude spectra of an FM signal, normalized with respect to the carrier amplitude, for the case of sinusoidal modulation of varying frequency and fixed amplitude. Only the spectra for positive frequencies are shown.

Transmission Bandwidth of FM Signals

- In theory, an FM signal contains an infinite number of side frequencies so that the bandwidth required to transmit such a signal is similarly infinite in extent.
- In practice, however, we find that the FM signal is effectively limited to a finite number of significant side frequencies compatible with a specified amount of distortion.
- Consider the case of an FM signal generated by a single-tone modulating wave of frequency f_m.
  - In such an FM signal, *the side frequencies that are separated from the carrier frequency f_c by an amount greater than the frequency deviation Δf decrease rapidly toward zero*, so that the bandwidth always exceeds the total frequency excursion, but nevertheless is limited.

Transmission Bandwidth of FM Signals

- We may thus define an approximate rule for the transmission bandwidth of an FM signal generated by a single-tone modulating signal of frequency f_m as follows:

$$B_T \simeq 2\Delta f + 2 f_m = 2\Delta f \left(1 + \frac{1}{\beta}\right) \qquad (4.38)$$

$$\boxed{\begin{array}{l} \text{Large } \beta \to B_T \simeq 2\Delta f \\ \text{Small } \beta \to B_T \simeq 2 f_m \end{array}}$$

This empirical relation is known as *Carson's rule*.

- For a more accurate assessment of the bandwidth requirement of an FM signal, we may thus define the transmission bandwidth of an FM wave as the separation between the two frequencies beyond which none of the side frequencies is greater than 1% of the carrier amplitude obtained when the modulation is removed.

Chapter 4.4
Phase-locked Loop

Wireless Information Transmission System Lab.
Institute of Communications Engineering
National Sun Yat-sen University

4.4 Phase-Locked Loop

- The phase-locked loop (PLL) is a negative feedback system, the operation of which is closely linked to frequency modulation.

- It can be used for synchronization, frequency division/multiplication, frequency modulation, and indirect frequency demodulation.

- Basically, the phase-locked loop consists of three major components: a *multiplier*, a *loop filter* (low-pass filter), and a *voltage-controlled oscillator* (VCO) connected together in the form of a feedback loop, as in Figure 4.16.

- The VCO is a sinusoidal generator whose frequency is determined by a voltage applied to it from an external source.

4.4 Phase-Locked Loop

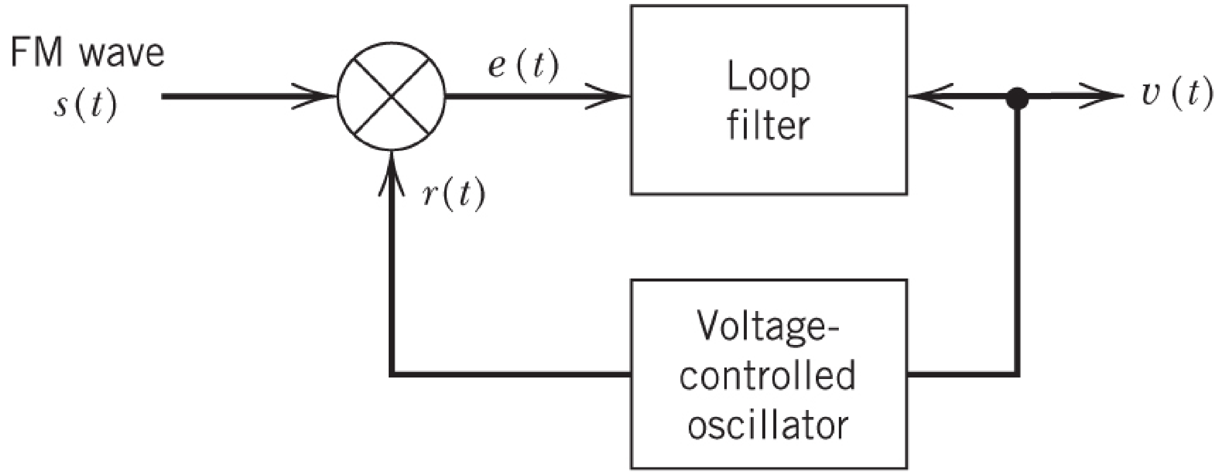


FIGURE 4.16 Phase-locked loop.

- We assume that initially we have adjusted the VCO so that when the control voltage is zero, two conditions are satisfied:

1. The frequency of the VCO in precisely set at the unmodulated carrier frequency f_c.
2. The VCO output has a 90-degree phase-shift with respect to the unmodulated carrier wave.

4.4 Phase-Locked Loop

- Suppose then that the input signal applied to the phase-locked loop is an FM signal defined by

$$s(t) = A_c \sin\left[2\pi f_c t + \phi_1(t)\right] \tag{4.59}$$

where A_c is the carrier amplitude and $\phi_1(t) = 2\pi k_f \int_0^t m(\tau)d\tau.$ (4.60)

- Let the VCO output in the phase-locked loop be defined by

$$r(t) = A_v \cos\left[2\pi f_c t + \phi_2(t)\right] \tag{4.61}$$

where A_v is the amplitude. With a control voltage $v(t)$ applied to a VCO input, the angle $\phi_2(t)$ is related to $v(t)$ by the integral

$$\phi_2(t) = 2\pi k_v \int_0^t \upsilon(t)\,dt \tag{4.62}$$

where k_v is the *frequency sensitivity* of the VCO, measured in Hertz per volt.

4.4 Phase-Locked Loop

- The object of the phase-locked loop is to generate a VCO output $r(t)$ that has the same phase angle (except for the fixed difference of 90 degrees) as the input FM signal $s(t)$.

- The time-varying phase angle $\phi_1(t)$ characterizing $s(t)$ may be due to modulation by a message signal $m(t)$ as in Eq. (4.60), in which case we wish to recover $\phi_1(t)$ in order to estimate $m(t)$.

- In other applications of the phase-locked loop, the time-varying phase angle $\phi_1(t)$ of the incoming signal $s(t)$ may be an unwanted phase shift caused by fluctuations in the communication channel; in this latter case, we wish to track $\phi_1(t)$ so as to produce a signal with the same phase angle for the purpose of *coherent detection* (*synchronous demodulation*).

Nonlinear Model of the PLL

- According to Figure 4.16, the incoming FM signal $s(t)$ and the VCO output $r(t)$ are applied to the multiplier, producing two components:

1. A high- frequency component, represented by the *double- frequency* term

$$k_m A_c A_v \sin\left[4\pi f_c t + \phi_1(t) + \phi_2(t)\right]$$

2. A low- frequency component, represented by the *difference- frequency* term

$$k_m A_c A_v \sin\left[\phi_1(t) - \phi_2(t)\right]$$

where k_m is the *multiplier gain*, measured in volt^{-1}.

- *The loop filter in the phase-locked loop is a low-pass filter*, and its response to the high- frequency component will be negligible.

Nonlinear Model of the PLL

- Therefore, discarding the high-frequency component (i.e., the double- frequency term), the input to the loop filter is reduced to

$$e(t) = k_m A_c A_\upsilon \sin\left[\phi_e(t)\right] \tag{4.63}$$

where $\phi_e(t)$ is the *phase error* defined by

$$\phi_e(t) = \phi_1(t) - \phi_2(t)$$

$$= \phi_1(t) - 2\pi k_\upsilon \int_0^t \upsilon(\tau) d\tau \tag{4.64}$$

- The loop filter operates on the input $e(t)$ to produce an output $v(t)$ defined by the convolution integral

$$\upsilon(t) = \int_{-\infty}^{\infty} e(\tau) h(t-\tau) d\tau \tag{4.65}$$

where $h(t)$ is the impulse response of the loop filter.

Linear Model of the PLL

- When the phase error $\phi_e(t)$ is zero, the phase-locked loop is said to be in phase-lock. When $\phi_e(t)$ is at all times small compared with one radian, we may use the approximation
$$\sin\left[\phi_e(t)\right] \simeq \phi_e(t) \tag{4.68}$$
which is accurate to within 4 percent for $\phi_e(t)$ less than 0.5 radians.
- We may represent the phase-locked loop by the linearized model shown in Figure 4.18*a*.

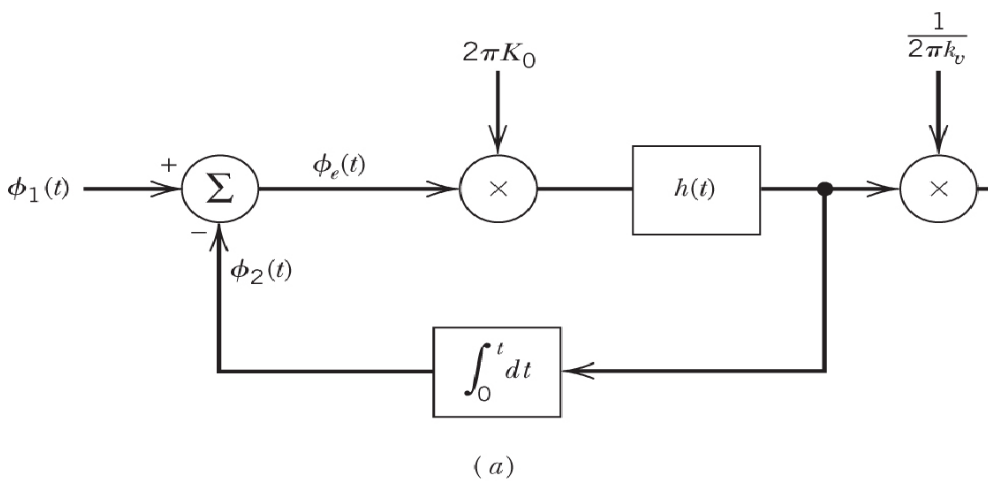


Figure 4.18 Models of the phase-locked loop. (*a*)Linearized model.