Chapter2
Fourier Theory and Communication Signals

Wireless Information Transmission System Lab.
Institute of Communications Engineering
National Sun Yat-sen University

Contents

Chapter 2.1
Introduction

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Chapter 2.1 Introduction

◇ We identify *deterministic signals* as a class of signals whose waveforms are defined exactly as functions of time.

◇ In this chapter we study the mathematical description of such signals using the Fourier transform that provides the link between the *time-domain* and *frequency-domain* descriptions of signal.

◇ Another related issue that we study in this chapter is the representation of *linear time-invariant* systems. Filters of different kinds and certain communication channels are important examples of this class of systems.

Chapter 2.2
The Fourier Transform

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◊ *Fourier transform* is defined as

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi ft) \, dt$$

   ◊ $g(t)$ denote a <u>nonperiodic</u> <u>deterministic</u> signal.

   ◊ $j = \sqrt{-1}$.

   ◊ variable f denotes *frequency* and t denotes time.

◊ *Inverse Fourier transform* is defined as

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft) \, df$$

Chapter 2.2 The Fourier Transform

◇ We have used a lowercase letter to denote the time function and a uppercase letter to denote the corresponding frequency function. The functions $g(t)$ *and* $G(f)$ are said to constitute a *Fourier-transform pair*.

◇ For the Fourier transform of a signal $g(t)$ to exist, it is sufficient, but not necessary, that $g(t)$ satisfies three sufficient conditions known collectively as *Dirichlet's conditions*:

  ◇ The function $g(t)$ is single-valued, with a finite number of maxima and minima in any finite time interval.

  ◇ The function $g(t)$ has finite number of discontinuities in any finite time interval.

  ◇ The function $g(t)$ is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |g(t)| \, dt < \infty$$

◇ We may safely ignore the question of the existence of the Fourier transform of a time function when it is an accurately specified description of a physically realizable signal.

◇ Physical realizability is a sufficient condition for the existence of a Fourier transform.

◇ All energy signals are Fourier transformable.

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| g(t) \right|^2 dt$$

Power Signal: $0 < P < \infty$

◇ *Energy signals*: $\int_{-\infty}^{\infty} \left| g(t) \right|^2 dt < \infty; \quad (P = 0)$

$(E = \infty)$

◇ *Plancherel's theorem*: if a time function $g(t)$ is such that the value of the energy $\int_{-\infty}^{\infty} \left| g(t) \right|^2 dt < \infty$ is defined and finite, then the Fourier transform $G(f)$ of the function $g(t)$ exists and

$$\lim_{A \to \infty} \left[\int_{-\infty}^{\infty} \left| g(t) - \int_{-A}^{A} G(f) \exp(j2\pi ft) df \right|^2 dt \right] = 0.$$

Chapter 2.2 The Fourier Transform

◇ Notations

  ◇ time t measured in *second* (s)

  ◇ frequency f measured in *Hertz* (Hz)

  ◇ *angular frequency* $\omega = 2\pi f$ (radians per second, rad/s).

  ◇ A convenient shorthand notation for the transform relations:

    ◇ Fourier transformation

$$G(f) = F\big[g(t)\big]$$

$$\xrightarrow{\;g(t)\;} \boxed{F[\;]} \xrightarrow{\;G(f)\;}$$

    ◇ Inverse Fourier transformation

$$g(t) = F^{-1}\big[G(f)\big]$$

$$\xrightarrow{\;G(f)\;} \boxed{F^{-1}[\;]} \xrightarrow{\;g(t)\;}$$

    where $F[\;]$ and $F^{-1}[\;]$ play the roles of *linear operators*.

    ◇ *Fourier-transform pair* $\quad g(t) \rightleftharpoons G(f)$

◇ Continuous Spectrum

  ◇ By using the Fourier transform operation, a pulse signal $g(t)$ of <u>finite energy</u> is expressed as a continuous sum of exponential functions with frequencies in the interval $-\infty \sim \infty$. The amplitude of a component of frequency f is proportional to $G(f)$, where $G(f)$ is the Fourier transform of $g(t)$.

  ◇ At any frequency f, the exponential function $\exp(j2\pi ft)$ is weighted by the factor $\underline{G(f)df}$, which is the contribution of $G(f)$ in an infinitesimal interval df centered at the frequency f.

  ◇ We may express the function $g(t)$ in terms of the continuous sum of such infinitesimal components:

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)\,df$$

◊ In general, the Fourier transform $G(f)$ is a complex function of frequency f :

$$G(f) = |G(f)| \exp\left[j\theta(f)\right]$$

$|G(f)|$ is called the continuous <u>amplitude spectrum</u> of $g(t)$

$\theta(f)$ is called the continuous <u>phase spectrum</u> of $g(t)$

◊ If $g(t)$ is a <u>real-valued</u> function of t, then

  ◊ $G(-f) = G^*(f)$

  ◊ $|G(-f)| = |G^*(f)| = |G(f)|$: an even function of f

  ◊ $\Theta(-f) = -\Theta(f)$: an odd function of f

  ◊ The spectrum of a real-valued signal exhibits *conjugate symmetry*.

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

$$G^*(f) = \left(\int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt\right)^* = \int_{-\infty}^{\infty} g(t) e^{j2\pi ft} dt = G(-f)$$

$$G(-f) = G^*(f) = |G(f)| \exp\left[-j\theta(f)\right]$$
$$G(-f) = |G(-f)| \exp\left[j\theta(-f)\right]$$
$$= |G(f)| \exp\left[-j\theta(f)\right]$$
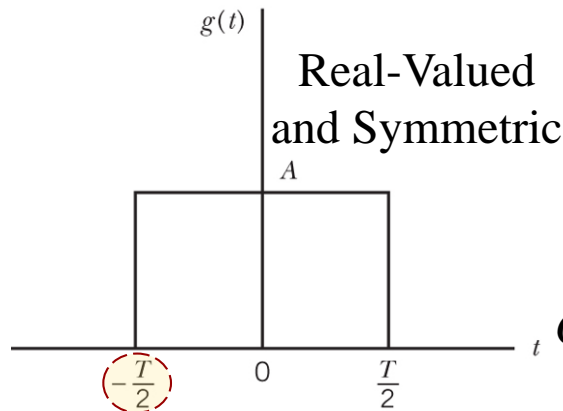
◇ [Example 2.1] Rectangular Pulse

   ◇ Define a *rectangular function* of unit amplitude and unit duration:

$$\text{rect}(t) = \begin{cases} 1, & -\dfrac{1}{2} < t < \dfrac{1}{2} \\ 0, & |t| \geq \dfrac{1}{2} \end{cases}$$

$g(t)$

Real-Valued
and Symmetric

A

$$g(t) = A\,\text{rect}\left(\dfrac{t}{T}\right)$$

$$A\,\text{rect}\left(\dfrac{t}{T}\right) \rightleftharpoons AT\,\text{sinc}(fT) \quad (2.10)$$

$-\dfrac{T}{2}$ 0 $\dfrac{T}{2}$

$$G(f) = \int_{-T/2}^{T/2} A\exp(-j2\pi ft)\,dt$$

$|G(f)|$

$$= \int_{-T/2}^{T/2} A\left(\cos(2\pi ft) - j\sin(2\pi ft)\right)dt$$

AT

Real-Valued
and Symmetric

$$= 2A\int_{0}^{T/2} \cos(2\pi ft)\,dt = 2A\,\dfrac{\sin(2\pi ft)}{2\pi f}\Bigg|_{0}^{\frac{T}{2}}$$

$-\dfrac{4}{T}$ $-\dfrac{3}{T}$ $-\dfrac{2}{T}$ $-\dfrac{1}{T}$ 0 $\dfrac{1}{T}$ $\dfrac{2}{T}$ $\dfrac{3}{T}$ $\dfrac{4}{T}$ f

$$= AT\left(\dfrac{\sin(\pi fT)}{\pi fT}\right) \equiv AT\,\text{sinc}(fT)$$

$$\text{sinc}(\lambda) \equiv \dfrac{\sin(\pi\lambda)}{\pi\lambda}$$
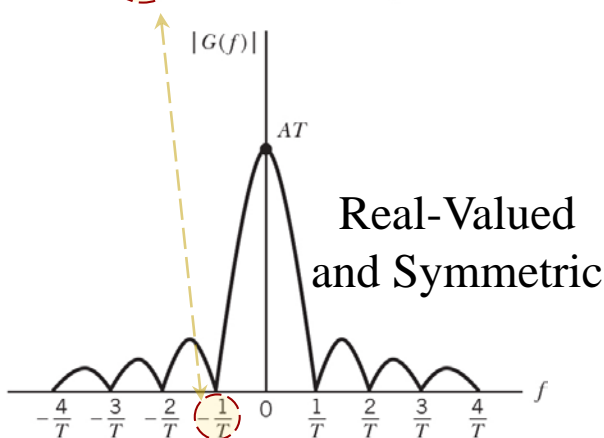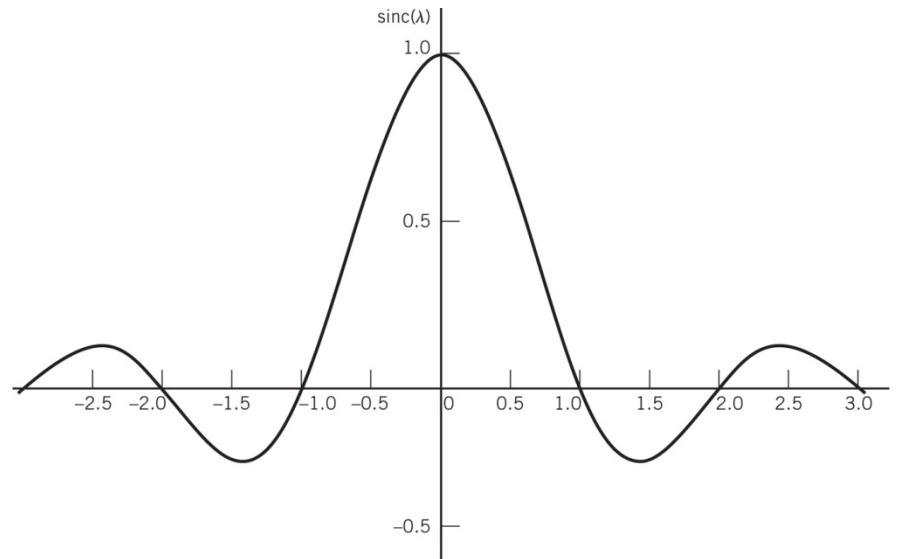
Chapter 2.2 The Fourier Transform

◇ sinc function

$$\text{sinc}(\lambda) \equiv \frac{\sin(\pi\lambda)}{\pi\lambda}$$

◇ As the pulse duration T is decreased, the first zero-crossing of the amplitude spectrum $|G(f)|$ moves up in frequency.

◇ The relationship between the time-domain and frequency-domain is an inverse one.

◇ A pulse, narrow in time, has a significant frequency description over a wide range of frequencies, and vice versa.

◇ [Example 2.2] Exponential Pulse

◇ A truncated form of a *decaying exponential pulse* is shown in the following figure
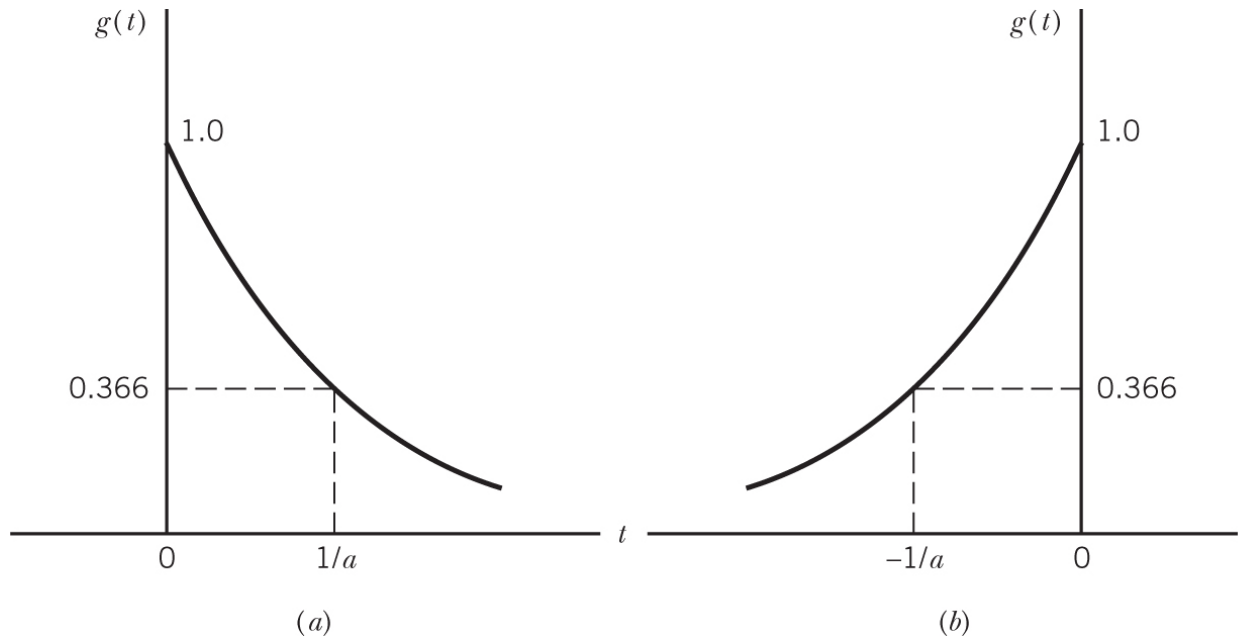
(a)Decaying exponential pulse. (b)Rising exponential pulse.

◇ [Example 2.2] Exponential Pulse(cont.)

   ◇ It is convenient to mathematically define the decaying exponential pulse using the *unit step function*.

   ◇ An *unit step function* is defined as:

$$u(t) = \begin{cases} 1, & t > 0 \\[2mm] \dfrac{1}{2}, & t = 0 \\[2mm] 0 & t < 0 \end{cases}$$

   ◇ Decaying exponential pulse of figure (a) can be expressed as

$$g(t) = \exp(-at)u(t)$$

the Fourier transform of this pulse is

$$G(f) = \int_0^\infty \exp(-at) \exp(-j2\pi ft) \, dt$$

$$= \int_0^\infty \exp\left[-t(a + j2\pi f)\right] dt = -\frac{\exp\left[-t(a + j2\pi f)\right]}{(a + j2\pi f)} \Bigg|_0^\infty$$

$$= \frac{1}{a + j2\pi f}$$

the Fourier-transform pair for the decaying exponential pulse of figure (a) is therefore

$$\exp(-at) u(t) \rightleftharpoons \frac{1}{a + j2\pi f}$$

◇ [Example 2.2] Exponential Pulse(cont.)

  ◇ Rising exponential pulse of Fig. (b)

$$g(t) = \exp(at)u(-t) \qquad \boxed{\exp(at)u(-t) \rightleftharpoons \frac{1}{a - j2\pi f}}$$

$$G(f) = \int_{-\infty}^{0} \exp(at)\exp(-j2\pi ft)\,dt$$

$$= \int_{-\infty}^{0} \exp\left[t(a - j2\pi f)\right]dt = \frac{1}{a - j2\pi f}$$

  ◇ The decaying and rising exponential pulses are both asymmetric functions of time t.

  ◇ Their Fourier transforms are therefore complex valued.

  ◇ Truncated decaying and rising exponential pulses have the same amplitude spectrum, but the phase spectrum of the one is the negative of that of the other.

Fourier-transform Pairs

Time Function	Fourier Transform
$\text{rect}\left(\dfrac{t}{T}\right)$	$T\,\text{sinc}(fT)$
$\text{sinc}(2Wt)$	$\dfrac{1}{2W}\text{rect}\left(\dfrac{f}{2W}\right)$
$\exp(-at)u(t),\quad a>0$	$\dfrac{1}{a+j2\pi f}$
$\exp(-a\lvert t\rvert),\quad a>0$	$\dfrac{2a}{a^2+(2\pi f)^2}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$\begin{cases}1-\dfrac{\lvert t\rvert}{T}, & \lvert t\rvert<T\\[2mm] 0, & \lvert t\rvert\geq T\end{cases}$	$T\,\text{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$\exp(-j2\pi ft_0)$
$\exp(j2\pi f_c t)$	$\delta(f-f_c)$
$\cos(2\pi f_c t)$	$\dfrac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\sin(2\pi f_c t)$	$\dfrac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)]$
$\text{sgn}(t)$	$\dfrac{1}{j\pi f}$
$\dfrac{1}{\pi t}$	$-j\,\text{sgn}(f)$
$u(t)$	$\dfrac{1}{2}\delta(f)+\dfrac{1}{j2\pi f}$
$\displaystyle\sum_{i=-\infty}^{\infty}\delta(t-iT_0)$	$\dfrac{1}{T_0}\displaystyle\sum_{n=-\infty}^{\infty}\delta\left(f-\dfrac{n}{T_0}\right)$

Notes: $u(t)$ = unit step function
$\delta(t)$ = Dirac delta function
$\text{rect}(t)$ = rectangular function
$\text{sgn}(t)$ = signum function
$\text{sinc}(t)$ = sinc function

	$g(t)$	$G(\omega)$
1.	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}\qquad a>0$
2.	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$
3.	$\lvert t\rvert$	$\dfrac{-2}{\omega^2}$
4.	$\delta(t)$	1
5.	1	$2\pi\,\delta(\omega)$
6.	$u(t)$	$\pi\,\delta(\omega)+\dfrac{1}{j\omega}$
7.	$\cos\omega_o t$	$\pi[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]$
8.	$\sin\omega_o t$	$j\pi[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)]$
9.	$\cos\omega_o t\,u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]+\dfrac{j\omega}{\omega_o^2-\omega^2}$
10.	$\sin\omega_o t\,u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_o)-\delta(\omega+\omega_o)]+\dfrac{\omega_o}{\omega_o^2-\omega^2}$
11.	$e^{-at}\sin\omega_o t\,u(t)$	$\dfrac{\omega_o}{(a+j\omega)^2+\omega_o^2}$
12.	$2B\,\text{sinc}\,(2Bt)$	$\Pi\left(\dfrac{\omega}{4\pi B}\right)$
13.	$\Pi\left(\dfrac{t}{\tau}\right)$	$\tau\,\text{sinc}\left(\dfrac{\omega\tau}{2\pi}\right)$
14.	$\begin{aligned}&1-\dfrac{\lvert t\rvert}{\tau}\cdots\lvert t\rvert<\tau\\ &\quad 0\cdots\lvert t\rvert>\tau\end{aligned}$	$\tau\,\text{sinc}^2\left(\dfrac{\omega\tau}{2\pi}\right)$
15.	$\text{sinc}^2\left(\dfrac{\omega_o t}{2\pi}\right)$	$\dfrac{2\pi}{\omega_o}\left(1-\dfrac{\lvert\omega\rvert}{\omega_o}\right)\quad\begin{aligned}&\lvert\omega\rvert\leq\omega_o\\&\lvert\omega\rvert>\omega_o\end{aligned}$
16.	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$
17.	$\displaystyle\sum_{k=-\infty}^{\infty}\delta(t-kT_o)$	$\omega_o\displaystyle\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_o)\qquad\omega_o=\dfrac{2\pi}{T_o}$
18.	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$

Chapter 2.3
Properties of The Fourier Transform

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◇ Summary of properties of the Fourier transform

Property	Mathematical Description
1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f),$ a and b are constants.
2. Time scaling	$g(at) \rightleftharpoons \dfrac{1}{\|a\|} G\left(\dfrac{f}{a}\right)$ where a is constant
3. Duality	If $g(t) \rightleftharpoons G(f)$ then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_c t) g(t) \rightleftharpoons G(f - f_c)$
6. Area under $g(t)$	$\displaystyle\int_{-\infty}^{\infty} g(t)\,dt = G(0)$

Property	Mathematical Description
7. Area under $G(f)$	$$g(0) = \int_{-\infty}^{\infty} G(f)\, df$$
8. Differentiation in the time domain	$$\frac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$$
9. Integration in the time domain	$$\int_{-\infty}^{t} g(\tau)\, d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) + \frac{G(0)}{2} \delta(f)$$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$ then $g^*(t) \rightleftharpoons G^*(-f)$
11. Multiplication in the time domain	$$g_1(t) g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda) G(f - \lambda)\, d\lambda$$
12. Convolution in the time domain	$$\int_{-\infty}^{\infty} g_1(\tau) g_2(t - \tau)\, d\tau \rightleftharpoons G_1(f) G_2(f)$$
13. Rayleigh's energy theorem	$$\int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$$

◇ Property 1：Linearity (Superposition)

Let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$. Then for all constants c_1 and c_2, we have

$$c_1 g_1(t) + c_2 g_2(t) \rightleftharpoons c_1 G_1 + c_2 G_2(f)$$

◇ Proof: the proof of this property follows simply from the linearity of the integrals defining $G(f)$ and $g(t)$.

◇ [Example 2.3] Combinations of Exponential Pulses

Consider a *double exponential pulse*

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases}$$

$$= \exp(-a|t|)$$

Symmetric in time domain.
Spectrum is real and symmetric.

This pulse may be viewed as the sum of a truncated decaying exponential pulse and a truncated rising exponential pulse.

$$G(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2}$$

$$\exp(-a|t|) \rightleftharpoons \frac{2a}{a^2 + (2\pi f)^2}$$

◇ [Example 2.3] Combinations of Exponential Pulses(cont.)

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 0, & t = 0 \\ -\exp(at), & t < 0 \end{cases}$$

$$\mathrm{sgn}(t) = \begin{cases} +1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

$$g(t) = \exp(-a|t|)\,\mathrm{sgn}(t)$$

◇ [Example 2.3] Combinations of Exponential Pulses(cont.)

$$F\left[\exp\left(-a|t|\right)\mathrm{sgn}\left(t\right)\right] = \frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f}$$

$$= -\frac{-j4\pi f}{a^2 + \left(2\pi f\right)^2}$$

$$\exp\left(-a|t|\right)\mathrm{sgn}\left(t\right) \rightleftharpoons \frac{-j4\pi f}{a^2 + \left(2\pi f\right)^2}$$

The Fourier transform is odd and purely imaginary.

◇ *In general, a real odd-symmetric time function has an odd and purely imaginary function as its Fourier transform.*

◇ Property 2：Time Scaling

Let $g(t) \rightleftharpoons G(f)$. Then $$g(at) \rightleftharpoons \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

◇ Proof：$$F\left[g(at)\right] = \int_{-\infty}^{\infty} g(at) \exp\left(-j2\pi ft\right) dt \qquad \tau = at \rightarrow t = \frac{\tau}{a}$$

$$\text{For } a > 0 : F\left[g(at)\right] = \frac{1}{a} \int_{-\infty}^{\infty} g(\tau) \exp\left[-j2\pi\left(\frac{f}{a}\right)\tau\right] d\tau = \frac{1}{a} G\left(\frac{f}{a}\right)$$

$$\text{For } a < 0 : F\left[g(at)\right] = \frac{1}{a} \int_{\infty}^{-\infty} g(\tau) \exp\left[-j2\pi\left(\frac{f}{a}\right)\tau\right] d\tau$$

$$= -\frac{1}{a} \int_{-\infty}^{\infty} g(\tau) \exp\left[-j2\pi\left(\frac{f}{a}\right)\tau\right] d\tau = -\frac{1}{a} G\left(\frac{f}{a}\right)$$

Q.E.D.

◇ Property 3：Duality

  ◇ If $g(t) \rightleftharpoons G(f)$, then $\boxed{G(t) \rightleftharpoons g(-f)}$

◇ Proof $\quad G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$$

$$\updownarrow \quad t \rightarrow f \quad f \rightarrow t \quad g(f) = \int_{-\infty}^{\infty} G(t) e^{j2\pi ft} dt$$

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi ft} dt = F\{G(t)\}$$

◇ [Example 2.4] $\quad A \operatorname{rect}\left(\dfrac{t}{T}\right) \rightleftharpoons AT \operatorname{sinc}(fT) \quad$ (2.10)

$$A \operatorname{sinc}(2Wt) \rightleftharpoons \frac{A}{2W} \operatorname{rect}\left(\frac{-f}{2W}\right) = \frac{A}{2W} \operatorname{rect}\left(\frac{f}{2W}\right)$$

◇ Property 4：Time Shifting

◇ If $g(t) \rightleftharpoons G(f)$, then $\boxed{g(t-t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)}$

◇ Proof ： Let $\tau = (t-t_0)$

$$F\left[g(t-t_0)\right] = \int_{-\infty}^{\infty} g(t-t_0)\exp(-j2\pi f t)\,dt$$

$$= \exp(-j2\pi f t_0)\int_{-\infty}^{\infty} g(\tau)\exp(-j2\pi f \tau)\,d\tau$$

$$= \exp(-j2\pi f t_0)G(f)$$

◇ The amplitude of $G(f)$ is unaffected by the time shift, but its phase is changed by the linear factor $-2\pi f t_0$.

◊ Property 5：Frequency Shifting (*Modulation Theorem*)

◊ If $g(t) \rightleftharpoons G(f)$, then $\boxed{\exp(j2\pi f_c t)g(t) \rightleftharpoons G(f - f_c)}$
where f_c is a real constant

◊ Proof：
$$F\left[\exp(j2\pi f_c t)g(t)\right] = \int_{-\infty}^{\infty} g(t)\exp\left[-j2\pi t(f - f_c)\right]dt$$
$$= G(f - f_c)$$

◊ Multiplication of a function by the factor exp(-2$\pi f_c t$) is equivalent to shifting its Fourier transform in the positive direction by the amount f_c.

◇ [Example 2.5]   Radio Frequency (RF) Pulse

Consider the pulse signal $g(t)$ shown in figure (a) which consists of a sinusoidal wave of amplitude A and frequency f_c, extending in duration from $t = -T/2$ to $t = T/2$. This signal is sometimes referred to as an *RF pulse* when the frequency f_c falls in the radio-frequency band. The signal $g(t)$ of figure (a) may be expressed mathematically as follows

$$g(t) = A \operatorname{rect}\left(\frac{t}{T}\right) \cos(2\pi f_c t)$$

(a)
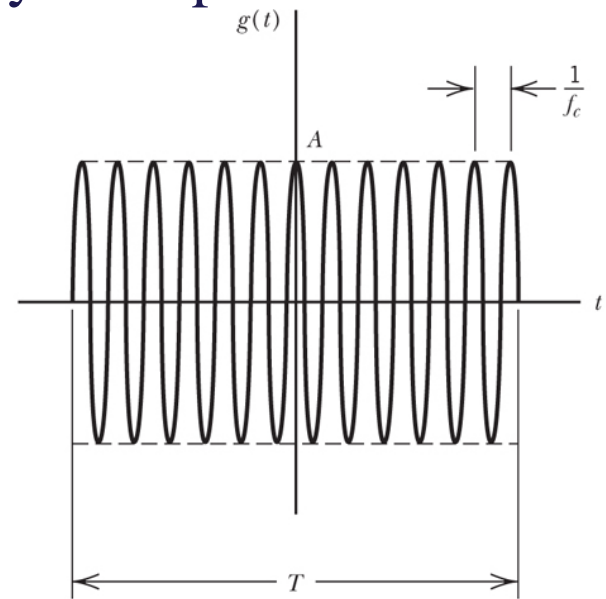
◊ [Example 2.5]   Radio Frequency (RF) Pulse (cont.)

we note that

$$\cos\left(2\pi f_c t\right) = \frac{1}{2}\left[\exp\left(j2\pi f_c t\right) + \exp\left(-j2\pi f_c t\right)\right]$$

applying the *frequency-shifting property* to the Fourier-transform pair, we get the desired result

$$G(f) = \frac{AT}{2}\left\{\operatorname{sinc}\left[T\left(f - f_c\right)\right] + \operatorname{sinc}\left[T\left(f + f_c\right)\right]\right\}$$

in the special case of $f_c T \gg 1$, we may use the approximate result

$$G(f) \simeq \begin{cases} \dfrac{AT}{2}\operatorname{sinc}\left[T\left(f - f_c\right)\right], & f > 0 \\ 0, & f = 0 \\ \dfrac{AT}{2}\operatorname{sinc}\left[T\left(f + f_c\right)\right], & f < 0 \end{cases}$$

◇ [Example 2.5]   Radio Frequency (RF) Pulse (cont.)

The amplitude spectrum of the RF pulse is shown in figure (b). This diagram, in relation to figure in page 12, clearly illustrates the frequency-shifting property of the Fourier transform.

(b)
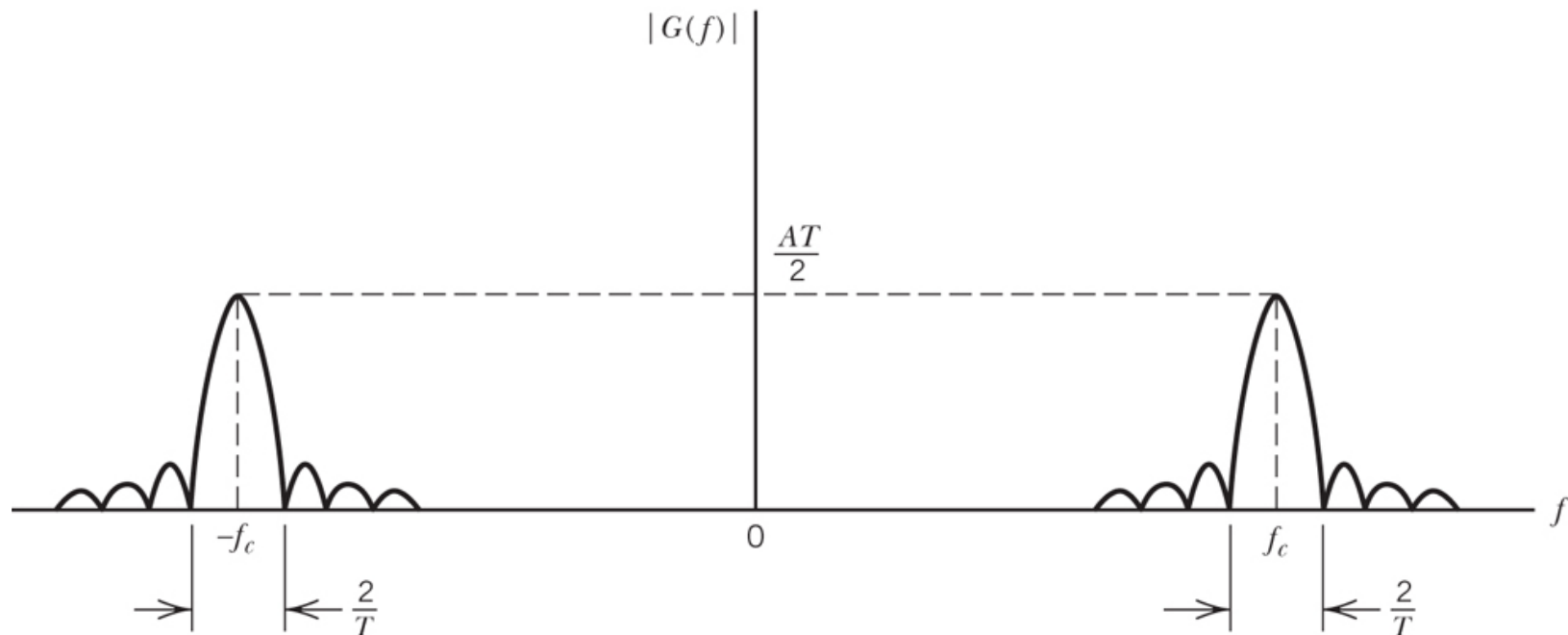
◇ Property 6：Area Under $g(t)$

If $g(t) \rightleftharpoons G(f)$, then $\boxed{\int_{-\infty}^{\infty} g(t)\,dt = G(0)}$ Physical meaning for G(0)?

That is, the area under a function $g(t)$ is equal to the value of its Fourier-transform $G(f)$ at $f = 0$. This result can be obtained by putting $f = 0$ in the formula of Fourier transform.

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)\,dt$$

◇ Property 7：Area Under $G(f)$

If $g(t) \rightleftharpoons G(f)$, then $\boxed{g(0) = \int_{-\infty}^{\infty} G(f)\,df}$

That is, the value of a function $g(t)$ at $t = 0$ is equal to the area under its Fourier-transform $G(f)$. This result can be obtained by putting $t = 0$ in the formula of inverse Fourier transform.

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)\,df$$

◇ Property 8：Differentiation in the Time Domain

Let $g(t) \rightleftharpoons G(f)$, and assume that the first derivative of $g(t)$ is Fourier transformable. Then

$$\frac{d}{dt} g(t) \rightleftharpoons j2\pi f \, G(f) \qquad (2.31)$$

That is, <u>differentiation of a time function $g(t)$ has the effect of multiplying its Fourier transform $G(f)$ by the factor $j2\pi f$</u>. If we assume that the Fourier transform of the higher-order derivative exists, then

$$\frac{d^n}{dt^n} g(t) \rightleftharpoons (j2\pi f)^n G(f)$$

◇ Proof：This result is obtained by taking the first derivative of both sides of the integral defining the inverse Fourier transform.

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft) df$$

◇ [Example 2.6]   Gaussian Pulse

  ◇ We will derive the particular form of a pulse signal that has the same mathematical form as its own Fourier transform.

  ◇ By differentiating the formula for the Fourier transform $G(f)$ with respect to f, we have

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi ft) dt$$

$$-j2\pi t g(t) \rightleftharpoons \frac{d}{df} G(f)$$

  ◇ Add (2.31) $\dfrac{d}{dt} g(t) \rightleftharpoons j2\pi f\, G(f)$ plus j times $-j2\pi t g(t) \rightleftharpoons \dfrac{d}{df} G(f)$

$$\frac{dg(t)}{dt} + 2\pi t g(t) \rightleftharpoons j\left[\frac{dG(f)}{df} + 2\pi f G(f) \right]$$

◇ If $\dfrac{dg(t)}{dt} = -2\pi t g(t)$, then $\dfrac{dG(f)}{df} = -2\pi f G(f)$

◇ [Example 2.6]   Gaussian Pulse (cont.)

◇ Since the pulse signal $g(t)$ and its Fourier transform $G(f)$ satisfy the same differential equation, they are the same function, i.e. $G(f) = g(f)$, where $g(f)$ is obtained from $g(t)$ by substituting f for t.

◇ Since $\dfrac{dg(t)}{dt} = -2\pi t g(t)$, we can obtain $g(t) = \exp\left(-\pi t^2\right)$

◇ This pulse is called a *Gaussian Pulse*.

$$\int_{-\infty}^{\infty} \exp\left(-\pi t^2\right) dt = 1$$

$$\boxed{\exp\left(-\pi t^2\right) \rightleftharpoons \exp\left(-\pi f^2\right)}$$

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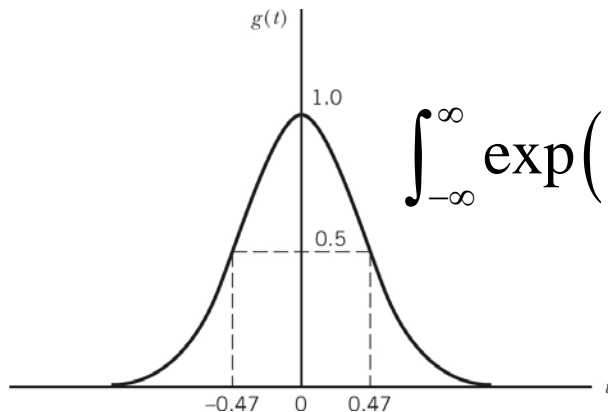
◊ Property 9：Integration in the Time Domain

◊ Let $g(t) \rightleftharpoons G(f)$. Then provided that $\underline{G(0)=0}$, we have

$$\int_{-\infty}^{t} g(\tau) d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) \qquad (2.39)$$

◊ Proof:

$$g(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} g(\tau) d\tau\right]$$

◊ Applying the time-differentiation property of the Fourier transform

$$\frac{d}{dt} x(t) \rightleftharpoons j2\pi f \, X(f)$$

$$F\left[g(t)\right] = G(f) = j2\pi f \left\{ F\left[\int_{-\infty}^{t} g(\tau) d\tau\right]\right\}$$

◇ [Example 2.7]   Triangular Pulse

   ◇ consider the *doublet pulse* $g_1(t)$ shown in Fig. (a). By integrating this pulse with respect to time, we obtain the *triangular pulse* $g_2(t)$.
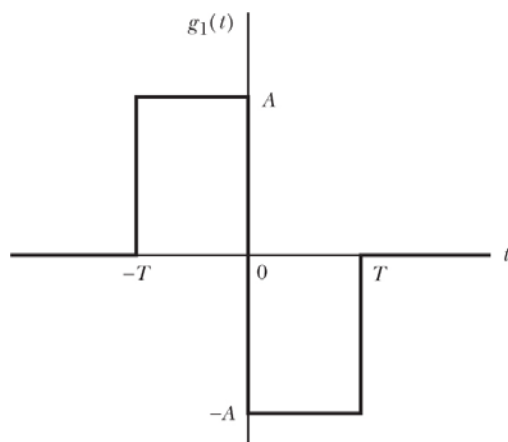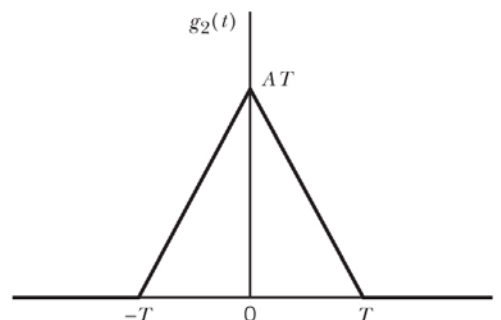
(a)   (b)

   ◇ The doublet pulse of figure (a) is real and odd-symmetric and its Fourier transform is therefore odd and purely imaginary.

   ◇ The triangular pulse of figure (b) is real and symmetric and its Fourier transform is therefore symmetric and purely real.

◇ [Example 2.7]　　Triangular Pulse (cont.)

　◇ $g_1(t)$ consists of two rectangular pulse:

　　◇ amplitude A, defined for the interval　$-T \le t \le 0$

　　　◇ Fourier transform:　$AT\,\text{sinc}(fT)\exp(j\pi fT)$

　　◇ amplitude $-A$, defined for the interval　$0 \le t \le T$

　　　◇ Fourier transform:　$-AT\,\text{sinc}(fT)\exp(-j\pi fT)$

　◇ Invoking the linearity property of the Fourier transform of $g_1(t)$

$$G_1(f) = AT\ \text{sinc}(fT)\big[\exp(j\pi fT) - \exp(-j\pi fT)\big]$$

$$= 2jAT\ \text{sinc}(fT)\sin(\pi fT) \qquad G_1(0) = 0$$

$$G_2(f) = \frac{1}{j2\pi f}G_1(f) = AT\frac{\sin(\pi fT)}{\pi f}\text{sinc}(fT)$$

$$= AT^2\text{sinc}^2(fT)$$

◇ Property 10：Conjugate Functions

  ◇ If $g(t) \rightleftharpoons G(f)$, then for a complex-valued time function $g(t)$ we have

$$g^*(t) \rightleftharpoons G^*(-f)$$

◇ Proof：
$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft) df$$

  ◇ taking the complex conjugates of both sides yields

$$g^*(t) = \int_{-\infty}^{\infty} G^*(f) \exp(-j2\pi ft) df$$

  ◇ Replacing f with $-f$ gives

$$g^*(t) = -\int_{\infty}^{-\infty} G^*(-f) \exp(j2\pi ft) df = \int_{-\infty}^{\infty} G^*(-f) \exp(j2\pi ft) df$$

  ◇ Corollary: $g^*(-t) \rightleftharpoons G^*(f)$

◇ [Example 2.8]　　Real and Imaginary Parts of a Time Function

◇

$$g(t) = \text{Re}\big[\, g(t) \,\big] + j \, \text{Im}\big[\, g(t) \,\big]$$

$$g^*(t) = \text{Re}\big[\, g(t) \,\big] - j \, \text{Im}\big[\, g(t) \,\big]$$

$$\text{Re}\big[\, g(t) \,\big] = \frac{1}{2}\big[\, g(t) + g^*(t) \,\big] \qquad \text{Im}\big[\, g(t) \,\big] = \frac{1}{2j}\big[\, g(t) - g^*(t) \,\big]$$

$$\text{Re}\big[\, g(t) \,\big] \rightleftharpoons \frac{1}{2}\big[\, G(f) + G^*(-f) \,\big]$$

$$\text{Im}\big[\, g(t) \,\big] \rightleftharpoons \frac{1}{2j}\big[\, G(f) - G^*(-f) \,\big]$$

$(\text{Im}[g(t)]=0)$

◇ If $g(t)$ is a real-valued time function, we have $G(f) = G^*(-f)$. In other words, $G(f)$ exhibits *conjugate symmetry*.

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◇ Property 11：Multiplication in the Time Domain

  ◇ Let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$. Then

$$g_1(t)\,g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)\,G_2(f-\lambda)\,d\lambda$$

◇ Proof：Let's denote $g_1(t)\,g_2(t) \rightleftharpoons G_{12}(f)$

$$G_{12}(f) = \int_{-\infty}^{\infty} g_1(t)\,g_2(t)\exp(-j2\pi ft)\,dt$$

  ◇ For $g_2(t)$, we have: $\quad g_2(t) = \int_{-\infty}^{\infty} G_2(f')\exp(j2\pi f't)\,df'$

$$G_{12}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(t)\,G_2(f')\exp\left[-j2\pi(f-f')t\right]df'\,dt$$

  ◇ Define: $\quad \lambda = f - f'$

$$G_{12}(f) = \int_{-\infty}^{\infty}\left[G_2(f-\lambda)\int_{-\infty}^{\infty} g_1(t)\exp(-j2\pi\lambda t)\,dt\right]d\lambda$$

◇ The inner integral is recognized as $G_1(\lambda)$

$$G_{12}(f) = \int_{-\infty}^{\infty} G_1(\lambda) G_2(f-\lambda) d\lambda \qquad \text{Q.E.D.}$$

◇ This integral is known as the *convolution integral* expressed in the frequency domain, and the function $G_{12}(f)$ is referred to as the *convolution* of $G_1(f)$ and $G_2(f)$.

◇ *The multiplication of two signals in the time domain is transformed into the convolution of their individual Fourier transforms in the frequency domain*. This property is known as the *multiplication theorem*.

◇ Notation:

$$G_{12}(f) = G_1(f) * G_2(f)$$

$$g_1(t) g_2(t) \rightleftharpoons G_1(f) * G_2(f)$$

$$G_1(f) * G_2(f) = G_2(f) * G_1(f)$$

◇ Property 12：Convolution in the Time Domain

    ◇ Let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$, then

$$\int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau \rightleftharpoons G_1(f) G_2(f)$$

◇ Proof：

$$g_{12}(t) = \int_{-\infty}^{\infty} G_1(f) G_2(f) e^{j2\pi ft} df$$

$$= \int_{-\infty}^{\infty} G_1(f) \left[\int_{-\infty}^{\infty} g_2(u) e^{-j2\pi uf} du \right] e^{j2\pi ft} df$$

Let $\lambda = t - u$

$$= \int_{-\infty}^{\infty} \left[g_2(t-\lambda) \int_{-\infty}^{\infty} G_1(f) e^{j2\pi f\lambda} df \right] d\lambda$$

$$= \int_{-\infty}^{\infty} g_1(\lambda) g_2(t-\lambda) d\lambda$$

Q.E.D.

◇ We may thus state that *the convolution of two signals in the time domain is transformed into the multiplication of their individual Fourier transforms in the frequency domain*.

◇ This property is known as the *convolution theorem*.

◇ Property 11 and property 12 are the dual of each other.

◇ Shorthand notation for convolution:

$$g_1(t) * g_2(t) \rightleftharpoons G_1(f)G_2(f)$$

◇ Property 13：Rayleigh's Energy Theorem

(Parseval's or Plancharel's theorem)

◇ Let $g(t)$ be defined over the entire interval $-\infty < t < \infty$ and assume its Fourier transform $G(f)$ exists. If the energy of the signal satisfies

$$E = \int_{-\infty}^{\infty} \left| g(t) \right|^2 dt < \infty$$

then

$$\int_{-\infty}^{\infty} \left| g(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| G(f) \right|^2 df$$

◇ $/G(f)/^2$ is defined as the *energy spectral density* (valid for energy signal).

◇ For power signal, we define *power spectral density* $S(f)$:

$$P = \int_{-\infty}^{\infty} S(f) df = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| g(t) \right|^2 dt \quad \text{(deterministic signal)}$$

◇ Proof:

$$E = \int_{-\infty}^{\infty} \left| g(t) \right|^2 dt = \int_{-\infty}^{\infty} g*(t) \cdot g(t) dt$$

$$= \int_{-\infty}^{\infty} g*(t) \left[\int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \right] dt$$

$$= \int_{-\infty}^{\infty} G(f) \left[\int_{-\infty}^{\infty} g*(t) e^{j2\pi ft} dt \right] df$$

$$= \int_{-\infty}^{\infty} G(f) \left[\int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \right]^* df$$

$$= \int_{-\infty}^{\infty} G(f) G*(f) df$$

$$= \int_{-\infty}^{\infty} \left| G(f) \right|^2 df$$

◊ [Example 2.9]   Sinc Pulse (continued)

  ◊ Consider the sinc pulse $A \text{ sinc}(2Wt)$. The energy of this pulse equals

  $$E = A^2 \int_{-\infty}^{\infty} \text{sinc}^2 (2Wt) \, dt$$

  ◊ The integral in the right-hand side of this equation is rather difficult to evaluate.

  ◊ From example 2.4, the Fourier transform of the sinc pulse $A \text{ sinc}(2Wt)$ is equal to $(A/2W)\text{rect}(f/2W)$. Applying Rayleigh's energy theorem

  $$E = \left(\frac{A}{2W} \right)^2 \int_{-\infty}^{\infty} \text{rect}^2 \left(\frac{f}{2W} \right) df$$

  $$= \left(\frac{A}{2W} \right)^2 \int_{-W}^{W} df = \frac{A^2}{2W}$$

◊ Summary of Properties for Fourier Transform

$g(t)$	Real Valued	Symmetric	Asymmetric	Real Valued and Symmetric	Real Valued and Odd Symmetric
$G(f)$ Spectrum	Conjugate Symmetry	Real Valued	Purely Imaginary	Real Valued and Symmetric	Odd and Purely Imaginary

◊ Compression of a function in the time domain is equivalent to the expansion of its Fourier transform in the frequency domain, or vice versa.

Chapter 2.4
The Inverse Relationship between Time and Frequency

Wireless Information Transmission System Lab.
Institute of Communications Engineering
National Sun Yat-sen University

◇ The time-domain and frequency-domain description of a signal are <u>inversely related</u>:

- ◇ If the time-domain description of a signal is changed, then the frequency-domain description of the signal is changed in an inverse manner, and vice versa.

- ◇ If a signal is strictly limited in frequency, then the time-domain description of the signal will trail on indefinitely.

  - ◇ A signal is strictly limited in frequency or strictly *band limited* if its Fourier transform is exactly zero outside a finite band of frequencies.

- ◇ If a signal is strictly limited in time, then the spectrum of the signal is infinite in extent.

  - ◇ A signal is strictly limited in time if the signal is exactly zero outside a finite time interval.

◇ Bandwidth

  ◇ The *bandwidth* of a signal provides a measure of the *extent of significant spectral content of the signal for positive frequencies*.

    ◇ When the signal is strictly band limited, the bandwidth is well defined.

    ◇ When the signal is not strictly band-limited, there is no universally accepted definition of bandwidth.

  ◇ A signal is said to be *low-pass* if its significant spectral content is centered *around the origin*.

  ◇ A signal is said to be *band-pass* if its significant spectral content is centered around $\pm f_c$, where f_c is a nonzero frequency.

◇ Bandwidth (cont.)

  ◇ When the spectrum of a signal is symmetric with a *main lobe* bounded by well-defined *nulls*(i.e. frequencies at which the spectrum is zero), we may use the main lobe as the basis for defining the bandwidth of the signal.

  ◇ When the signal is <u>low-pass</u>, the bandwidth is defined as *one half the total width of the main spectral lobe*, since only one half of this lobe lies inside the positive frequency region.

  ◇ When the signal is <u>band-pass</u> with main spectral lobes centered around $\pm f_c$, where f_c is large, the bandwidth is defined as the *width of the main lobe for positive frequency*. This definition of bandwidth is called the *null-to-null bandwidth*.

◇ Bandwidth (cont.)

◇ 3-dB Bandwidth

◇ When the signal is <u>low-pass</u>, the *3-dB bandwidth* is defined as the separation between *zero frequency*, where the amplitude spectrum attains its peak value, and the *positive frequency* at which the amplitude spectrum drops to $1/\sqrt{2}$ of its peak value.

◇ When the signal is <u>band-pass</u>, centered at $\pm f_c$, the *3-dB bandwidth* is defined as the separation (along the positive frequency axis) between the two frequencies at which the amplitude spectrum of the signal drops to $1/\sqrt{2}$ of the peak value at f_c.

◇ Advantage : it can be read directly from a plot of the amplitude spectrum.

◇ Disadvantage: it may be misleading if the amplitude spectrum has slowly decreasing tails.
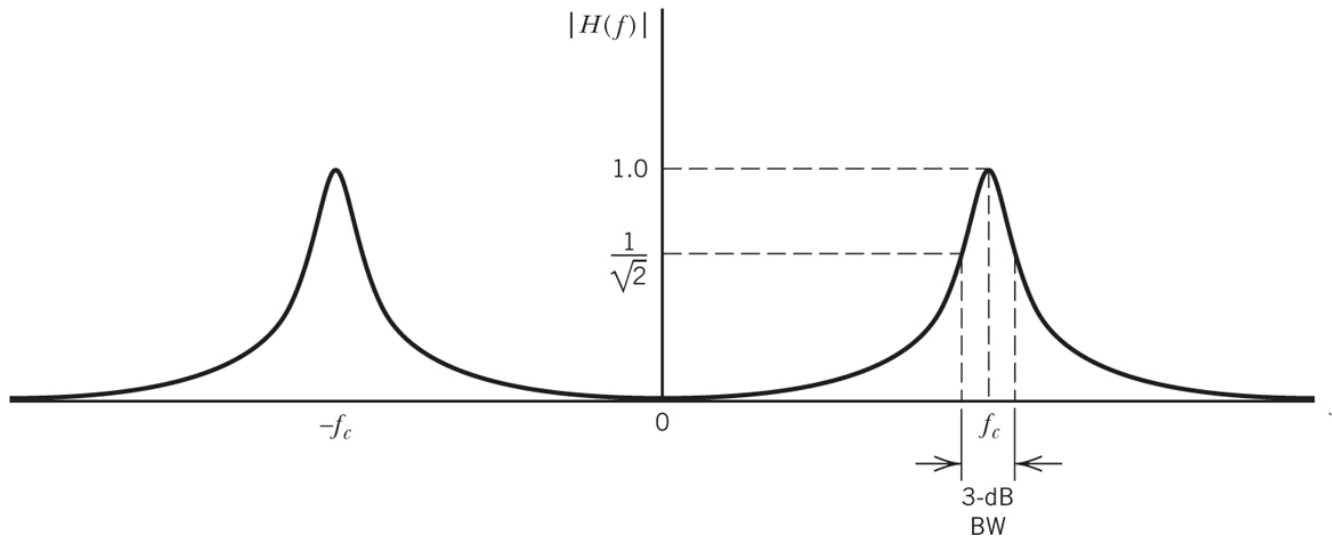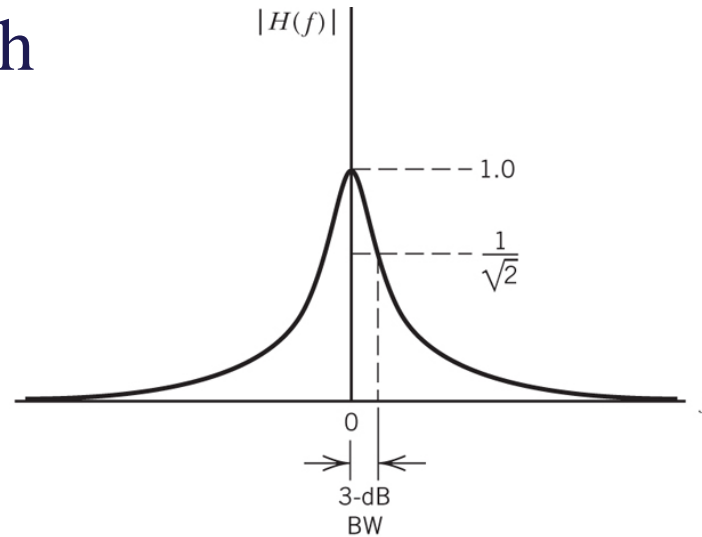
◇ 3-dB Bandwidth

◇ Root Mean Square (rms) Bandwidth

- ◇ _Root Mean Square_ (rms) bandwidth, defined as the square root of the second moment of a properly normalized form of the squared amplitude spectrum of the signal about a suitably chosen point.

- ◇ The rms bandwidth of a low-pass signal is formally defined as:

$$W_{\mathrm{rms}} = \left(\frac{\int_{-\infty}^{\infty} f^2 \left| G(f) \right|^2 df}{\int_{-\infty}^{\infty} \left| G(f) \right|^2 df} \right)^{\frac{1}{2}}$$

- ◇ An attractive feature of the rms bandwidth is that it lends itself more readily to mathematical evaluation than the other two definitions of bandwidth, but it is not as easily measurable in the laboratory.

◇ Time-Bandwidth product

  ◇ For any family of pulse signals (e.g. the exponential pulse) that differ in time scale, the product of the signal's duration and its bandwidth is always a constant, as shown by

$$(\text{duration}) \cdot (\text{bandwidth}) = \text{constant}$$

  ◇ The product is called the *time-bandwidth product* or *bandwidth-duration product*.

  ◇ If the duration of a pulse signal is decreased by reducing the time scale by a factor a, the frequency scale of the signal's spectrum, and therefore the bandwidth of the signal, is increased by the same factor a.

◇ Time-Bandwidth product (cont.)

  ◇ Consider the rms bandwidth. The corresponding definition for the *rms duration* is

$$T_{\text{rms}} = \left(\frac{\int_{-\infty}^{\infty} t^2 \left| g(t)^2 \right| dt}{\int_{-\infty}^{\infty} \left| g(t) \right|^2 dt} \right)^{\frac{1}{2}}$$

  ◇ The time-bandwidth product has the following form:

$$T_{\text{rms}} W_{\text{rms}} \geq \frac{1}{4\pi}$$

  ◇ Gaussian pulse satisfies this condition with the equality sign.

Chapter 2.5
Dirac Delta Function

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National Sun Yat-sen University

◊ The *Dirac delta function*, denoted by $\delta(t)$, is defined as having zero amplitude everywhere except at $t = 0$, where it is infinitely large in such a way that it contains unit area under its curve; i.e.

$$\delta(t) = 0, \ t \neq 0 \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

◊ The delta function $\delta(t)$ is an <u>even function of time t</u>.

◊ *Shifting property* of the delta function:

$$\int_{-\infty}^{\infty} g(t)\delta(t - t_0)\,dt = g(t_0)$$

◊ *Replication property* of the delta function: the convolution of any function with the delta function leaves that function unchanged.

$$g(t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau)\delta(t - \tau)\,d\tau = g(t)$$

Chapter 2.5 Dirac Delta Function

◊ Fourier transform of the delta function is given by

$$F\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t)\exp(-j2\pi ft)\,dt = 1 \qquad \boxed{\delta(t) \rightleftharpoons 1}$$

◊ This relation states that the spectrum of the delta function $\delta(t)$ extends uniformly over the entire frequency interval.

◊ We may view the delta function as *the limiting form of a pulse of unit area as the duration of the pulse approaches zero*.

$$A\,\text{rect}\left(\frac{t}{T}\right) \rightleftharpoons AT\,\text{sinc}(fT)$$

(a) (b)
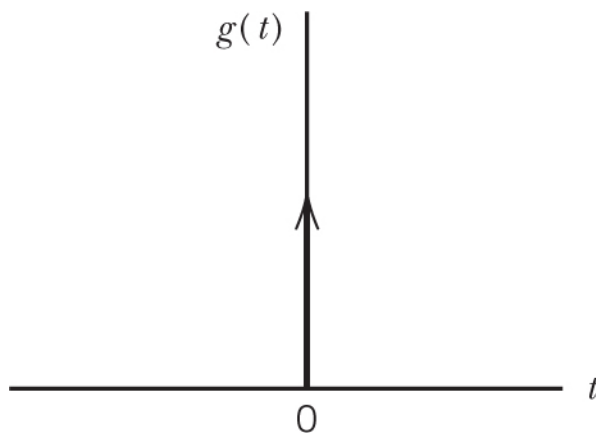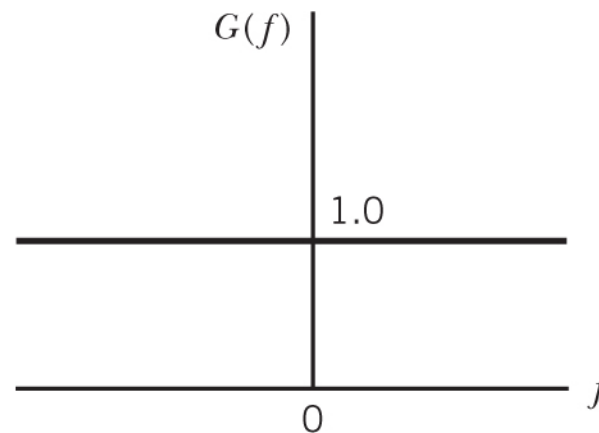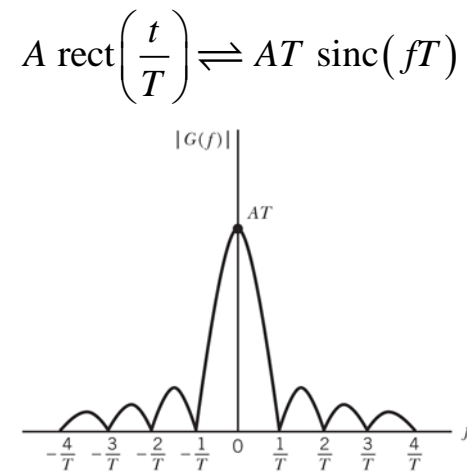
◇ Applications of the Delta Function

◇ DC Signal   If $g(t) \rightleftharpoons G(f)$ then $G(t) \rightleftharpoons g(-f)$ $1 \rightleftharpoons \delta(-f)$

◇ By applying the duality property to the Fourier-transform pair of $\delta(t) \rightleftharpoons 1$ and noting that the delta function is an even function, we obtain

$$1 \rightleftharpoons \delta(f)$$

◇ DC signal is transformed in the frequency domain into a delta function.

(a)   (b)
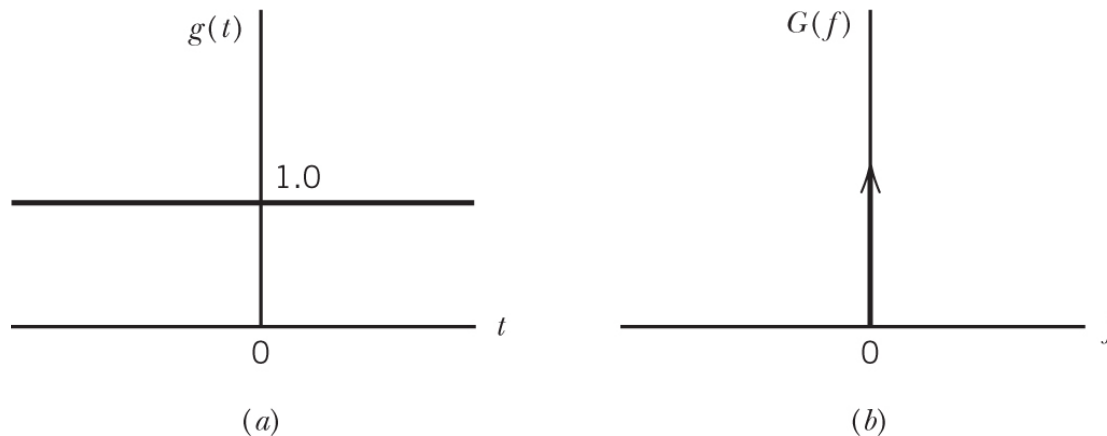
◇ Another definition for the delta function:

$$\int_{-\infty}^{\infty} \exp(-j2\pi ft)\,dt = \delta(f) \xrightarrow{\text{Delta functin is real.}} \boxed{\int_{-\infty}^{\infty} \cos(2\pi ft)\,dt = \delta(f)}$$

◇ Applications of the Delta Function

  ◇ Complex Exponential Function

$$1 \rightleftharpoons \delta(f) \quad \xrightarrow[\displaystyle \exp(j2\pi f_c t)g(t) \rightleftharpoons G(f-f_c)]{\text{Applying the frequency-shifting property:}} \quad \exp(j2\pi f_c t) \rightleftharpoons \delta(f-f_c)$$

  ◇ Sinusoidal Functions

    ◇ By using Euler's formula:

$$\cos(2\pi f_c t) = \frac{1}{2}\left[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)\right]$$

$$\cos(2\pi f_c t) \rightleftharpoons \frac{1}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right] \qquad \sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}\left[\delta(f-f_c) - \delta(f+f_c)\right]$$

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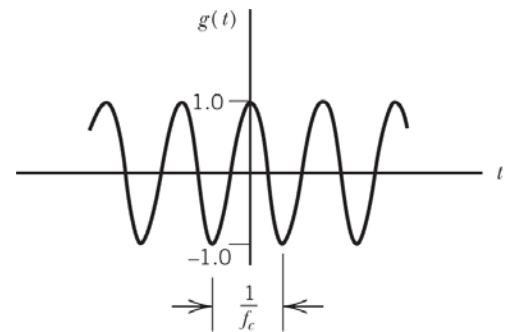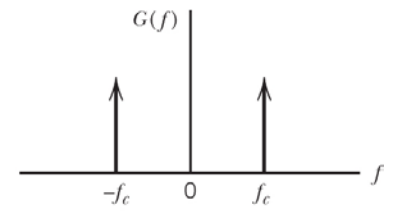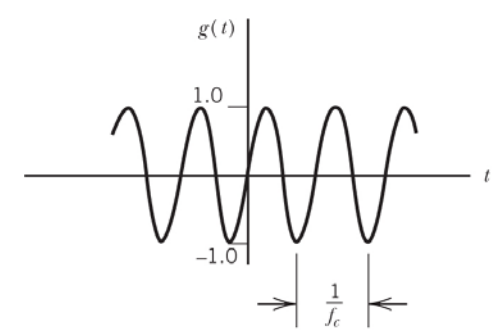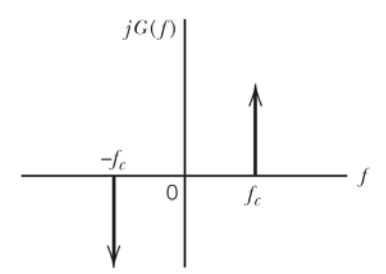
◇ **Applications of the Delta Function**

◇ Signum Function：sgn(t)

◇ Definition:

$$\operatorname{sgn}(t) = \begin{cases} +1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

◇ The signum function does not satisfy the Dirichlet conditions and does not have a Fourier transform.

◇ The signum function can be viewed as the limiting form of the antisymmetric double-exponential pulse as the parameter a approaches 0.

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 0, & t = 0 \\ -\exp(at), & t < 0 \end{cases}$$

(b)
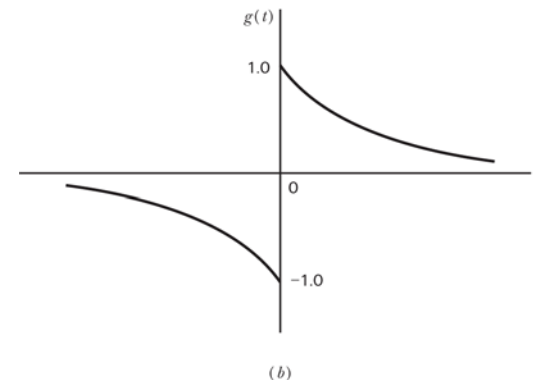
◊ Applications of the Delta Function

◊ Signum Function (cont.)

◊ From Example 2.3 for double exponential pulse $G(f) = \dfrac{-j4\pi f}{a^2 + (2\pi f)^2}$

$$F(\operatorname{sgn}(t)) = \lim_{a \to 0} \frac{-4j\pi f}{a^2 + (2\pi f)^2} = \frac{1}{j\pi f}$$

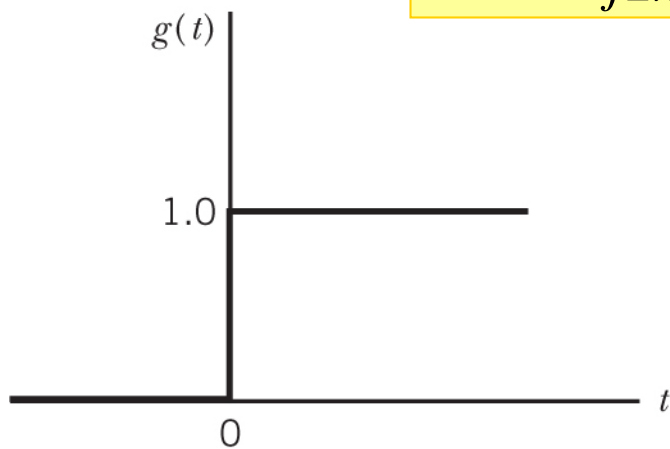$$\operatorname{sgn}(t) \rightleftharpoons \frac{1}{j\pi f}$$

(a)

(b)
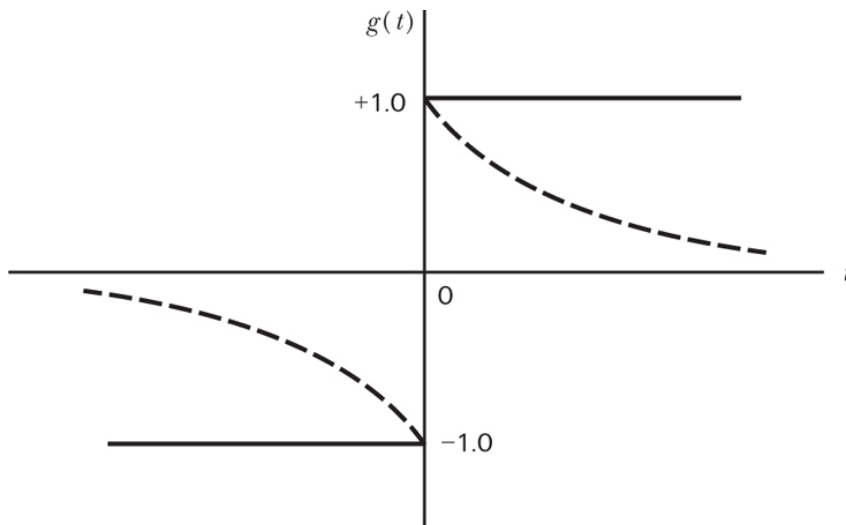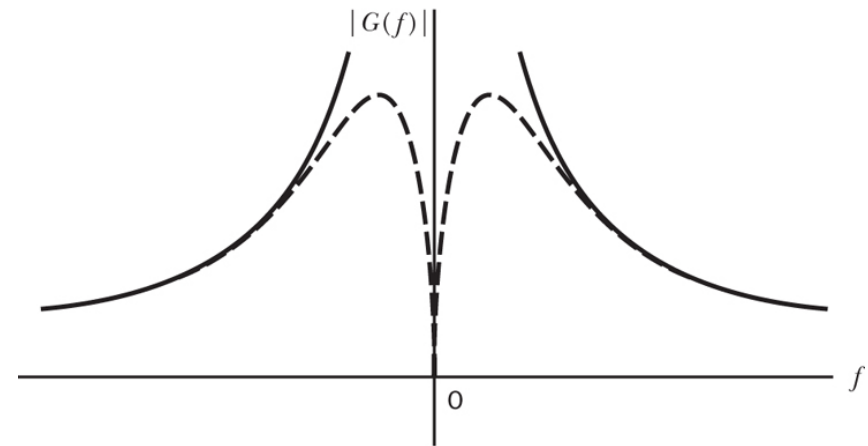
◇ Applications of the Delta Function

　◇ Unit Step Function

$$u(t) = \begin{cases} 1, & t > 0 \\ \dfrac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \qquad u(t) = \dfrac{1}{2}\big[\mathrm{sgn}(t) + 1\big]$$

　　◇ By using the linearity property of the Fourier transform and $1 \rightleftharpoons \delta(f)$

$$u(t) \rightleftharpoons \dfrac{1}{j2\pi f} + \dfrac{1}{2}\delta(f)$$

$g(t)$

1.0

0

(a)

$|G(f)|$

0

(b)
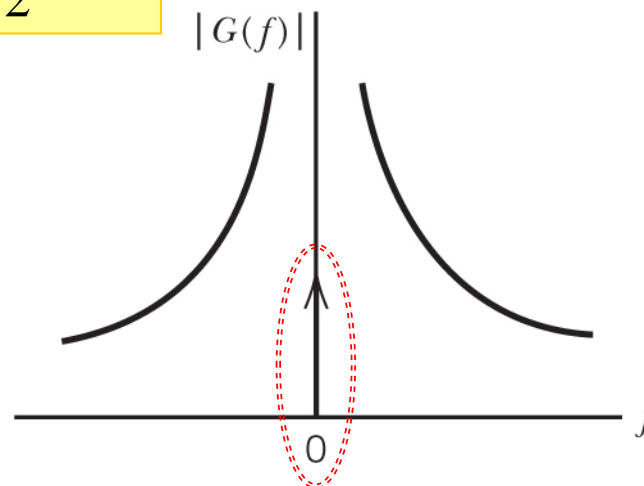
◇ Applications of the Delta Function

◇ Integration in the Time Domain (Revisited)

◇ Let

$$y(t) = \int_{-\infty}^{t} g(\tau) d\tau$$

◇ The integrated signal $y(t)$ can be viewed as the convolution of the original signal $g(t)$ and the unit step function $u(t)$, as shown by

$$y(t) = \int_{-\infty}^{\infty} g(\tau) u(t - \tau) d\tau = g(t) * u(t)$$

◇ The time-shifted unit step function $u(t - \tau)$ is defined by

$$u(t - \tau) = \begin{cases} 1, & \tau < t \\ \dfrac{1}{2}, & \tau = t \\ 0, & \tau > t \end{cases}$$

◊ Applications of The Delta Function (cont.)

   ◊ Integration in the Time Domain (Revisited) (cont.)

      ◊ The Fourier transform of $y(t)$ can be easily obtained:

$$Y(f) = G(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$$

      ◊ Since

$$G(f)\delta(f) = G(0)\delta(f)$$

$$Y(f) = \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f)$$

$$\int_{-\infty}^{t} g(\tau)\,d\tau \rightleftharpoons \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f)$$

      ◊ Eq. (2.39) is a special case of the above equation with $G(0)=0$.

Chapter 2.6
Fourier Transform of Periodic Signals

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◇ A periodic signal can be represented in terms of a Fourier transform provided that this transform is permitted to include delta functions.

◇ Consider a periodic signal $g_{T_0}(t)$ of period T_0:

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j2\pi nf_0 t\right)$$

where c_n is the complex Fourier coefficient defined by

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t) \exp\left(-j2\pi nf_0 t\right) dt$$

and f_0 is the *fundamental frequency* $f_0 = 1/T_0$.

◇ Let $g(t)$ be a pulse like function, which equals $g_{T_0}(t)$ over one period and is zero elsewhere; that is,

$$g(t) = \begin{cases} g_{T_0}(t), & -\dfrac{T_0}{2} \leq t \leq \dfrac{T_0}{2} \\ 0, & \text{elsewhere} \end{cases}$$

◇ $g_{T_0}(t)$ may now be expressed in terms of the function $g(t)$

$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_0)$$

◇ $g(t)$ is Fourier transformable and can be viewed as a *generating function*, which generates the periodic signal $g_{T_0}(t)$.

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t) \exp(-j2\pi n f_0 t)\, dt = f_0 \int_{-\infty}^{\infty} g(t) \exp(-j2\pi n f_0 t)\, dt \equiv f_0 G(nf_0)$$

where $G(nf_0)$ is the Fourier transform of $g(t)$ at the frequency nf_0.

◇ The formula for the reconstruction of the periodic signal $g_{T_0}(t)$ can be rewritten as:

$$c_n = f_0 G(nf_0)$$

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi nf_0 t) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0) \exp(j2\pi nf_0 t)$$

$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_0) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0) \exp(j2\pi nf_0 t)$$

The above equation is one form of *Poisson's sum formula*.

$$\exp(j2\pi f_c t) \rightleftharpoons \delta(f - f_c)$$

$$\sum_{m=-\infty}^{\infty} g(t - mT_0) \rightleftharpoons f_0 \sum_{n=-\infty}^{\infty} G(nf_0) \delta(f - nf_0) \qquad (2.88)$$

◇ *The Fourier transform of a periodic signal consists of delta functions occurring at integer multiples of the fundamental frequency $f_0 = 1/T_0$, including the origin, and that each delta function is weighted by a factor equal to the corresponding value of $G(nf_0)$.*

◇ The function $g(t)$, constituting one period of the periodic signal $g_{T_0}(t)$, has a <u>continuous spectrum</u> defined by $G(f)$.

◇ The periodic signal $g_{T_0}(t)$ has a <u>discrete spectrum</u>.

◇ *<u>Periodicity in the time domain has the effect of changing the frequency-domain description or spectrum of the signal into a discrete form defined at integer multiples of the fundamental frequency</u>.*

◇ [Example 2.11] Ideal Sampling Function

◇ An ideal sampling function, or *Dirac comb*, consists of an infinite sequence of uniformly spaced delta functions.

$$\delta_{T_0}(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$$

◇ The generating function $g(t)$ for the ideal sampling function $\delta_{T_0}(t)$ consists of the delta function $\delta(t)$. We therefore have $G(f)=1$ and $G(nf_0)=1$ for all n.

◇ Using Eq. (2.88) $\sum_{m=-\infty}^{\infty} g(t - mT_0) \rightleftharpoons f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0)$ yields

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) \rightleftharpoons f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

◇ The Fourier transform of a periodic train of delta functions, spaced T_0 seconds apart, consists of another set of delta functions weighted by the factor $f_0=1/T_0$ and regularly spaced f_0 Hz apart along the frequency axis.

◇ [Example 2.11] Ideal Sampling Function (cont.)

$\delta_{T_0}(t)$

$-\dfrac{4}{T}$

$-3T_0$ $-2T_0$ $-T_0$ 0 T_0 $2T_0$ $3T_0$ t

(a)

$F|\delta_{T_0}(t)|$

$-\dfrac{5}{T_0}$ $-\dfrac{4}{T_0}$ $-\dfrac{3}{T_0}$ $-\dfrac{2}{T_0}$ $-\dfrac{1}{T_0}$ 0 $\dfrac{1}{T_0}$ $\dfrac{2}{T_0}$ $\dfrac{3}{T_0}$ $\dfrac{4}{T_0}$ $\dfrac{5}{T_0}$ f

(b)
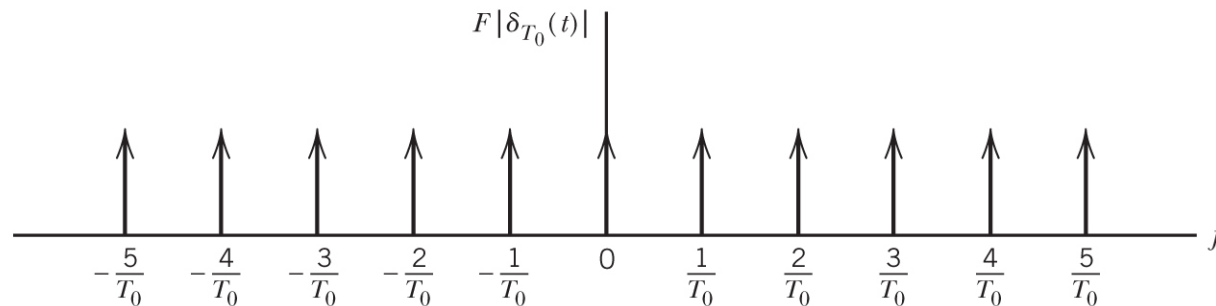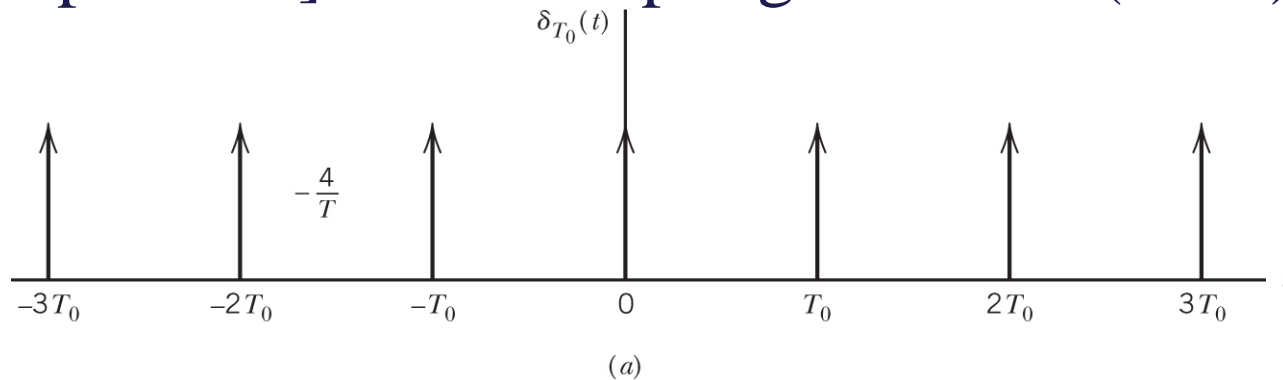
◇ From Poisson's sum formula: $\displaystyle\sum_{m=-\infty}^{\infty} g\left(t-mT_0\right)= f_0 \sum_{n=-\infty}^{\infty} G\left(nf_0\right)\exp\left(j2\pi nf_0 t\right)$

$$\sum_{m=-\infty}^{\infty} \delta\left(t-mT_0\right)= f_0 \sum_{n=-\infty}^{\infty} \exp\left(j2\pi nf_0 t\right)$$

Fourier Dual

$$\sum_{m=-\infty}^{\infty} \exp\left(j2\pi mfT_0\right)= f_0 \sum_{n=-\infty}^{\infty} \delta\left(f-nf_0\right)$$

Nyquist Sampling Theorem

◇ A band-limited signal of finite energy, which only has frequency components less than f_m Hertz, is completely described by specifying the values of the signal at instants of time separated by $1/2\,f_m$ seconds.

$$T_S \leq \frac{1}{2f_m} \quad \text{or sampling rate } f_S \geq 2f_m$$

◇ A band-limited signal of finite energy, which only has frequency components less than f_m Hertz, may be completely recovered from a knowledge of its samples taken at the rate of $2\,f_m$ samples per second.

◇ The sampling rate of $2f_m$ per second, for a signal bandwidth of f_m Hertz, is called the *Nyquist rate*; its reciprocal $1/2\,f_m$ (measured in seconds) is called the *Nyquist interval*.

$$x(t)$$

$$(a)$$

$$|X(f)|$$

$$-f_m \quad 0 \quad f_m$$

$$(b)$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$(c)$$

$$X_\delta(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$

$$(d)$$

$$x_s(t) = x(t)x_\delta(t)$$

$$(e)$$

$$|X_s(f)| \qquad X_s(f) = X(f) * X_\delta(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

$$(f)$$

Figure 2.7 Spectra for various sampling rates. (a) Sampled spectrum ($f_s > 2f_m$). (b) Sampled spectrum ($f_s < 2f_m$).

Chapter 2.7
Transmission of Signals Through Linear Systems

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◇ *System*: any physical device that produces an output signal in response to an input signal.

◇ *Excitation*: input signal.

◇ *Response*: output signal.

◇ In a *linear system*, the *principle of superposition* holds, i.e., <u>the response of a linear system to a number of excitations applied simultaneously is equal to the sum of the responses of the system when each excitation is applied individually.</u>

  ◇ Important examples: filters, communication channels.

◇ *Filter*: a frequency-selective device that is used to limit the spectrum of a signal to some band of frequencies.

◇ *Channel*: transmission medium that connects the transmitter and receiver of a communication system.
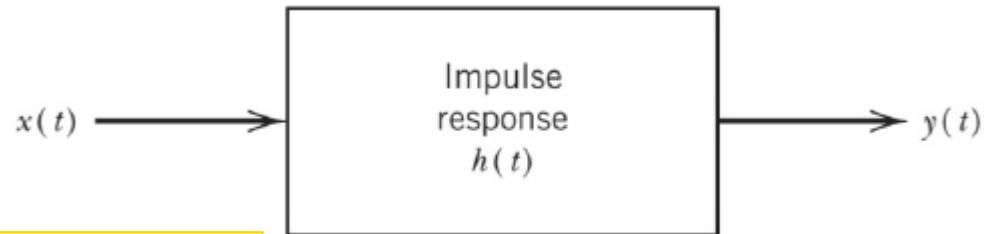
◇ Time Response

  ◇ In the time domain, a linear system is described in terms of its *impulse response*, which is defined as the response of the system (with zero initial conditions) to *a unit impulse* or *delta function* $\delta(t)$ applied to the input of the system.

  ◇ If the system is *time invariant*, then the shape of the impulse response is the same no matter when the unit impulse is applied to the system.

  ◇ *Convolution Integral*:

$x(t) \longrightarrow$ | Impulse response $h(t)$ | $\longrightarrow y(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = h(t) * x(t)$$

◇ Causality and Stability

  ◇ *Causal*: A system is said to be causal if it does not respond before the excitation is applied.

    ◇ For a *linear time-invariant* (LTI) system to be causal, the impulse response $h(t)$ must vanish for negative time, i.e. $h(t)=0$, $t<0$.

    ◇ A system operating in <u>real time</u> to be physically realizable, it must be causal.

    ◇ The system can be noncausal and yet physically realizable. (non-real-time).

  ◇ *Stable*: A system is said to be stable if the output signal is bounded for all bounded input signals.

    ◇ *Bounded input-bounded output* (BIBO) *stability criterion*.

    ◇ For a LTI system to be stable, the impulse response must be absolutely integrable, i.e.
    $$\int_{-\infty}^{\infty} |h(t)| dt < \infty \qquad (2.100)$$

◇ Frequency Response

   ◇ Consider a LTI system of impulse response $h(t)$ driven by a complex exponential input of unit amplitude and frequency f

$$x(t) = \exp(j2\pi ft)$$

   ◇ The response of the system is obtained as

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) \exp\left[j2\pi f(t-\tau)\right] d\tau$$

$$= \exp(j2\pi ft)\int_{-\infty}^{\infty} h(\tau)\exp(-j2\pi f\tau)d\tau$$

   ◇ *Transfer function* of the system is defined as the Fourier transform of its impulse response

$$H(f) \equiv \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt \qquad y(t) = H(f)\exp(j2\pi ft)$$

$$= H(f)x(t)$$

$$H(f) \equiv \left.\frac{y(t)}{x(t)}\right|_{x(t)=\exp(j2\pi ft)}$$

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◇ Frequency Response (cont.)

◇ Consider an arbitrary signal $x(t)$ applied to the system:

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df$$

or, equivalently, in the limiting form (a superposition of complex <u>exponentials</u> of incremental amplitude)

$$x(t) = \lim_{\substack{\Delta f \to 0 \\ f = k\Delta f}} \sum_{k=-\infty}^{\infty} X(f)\exp(j2\pi ft)\Delta f$$

◇ Because the system is linear, the response is:

$$y(t) = \lim_{\substack{\Delta f \to 0 \\ f = k\Delta f}} \sum_{k=-\infty}^{\infty} H(f)X(f)\exp(j2\pi ft)\,df \qquad \boxed{y(t) = H(f)x(t)}$$

$$= \int_{-\infty}^{\infty} H(f)X(f)\exp(j2\pi ft)\,df \qquad \boxed{y(t) = \int_{-\infty}^{\infty} Y(f)\exp(j2\pi ft)\,df}$$

$$\boxed{Y(f) = H(f)X(f)}$$

◊ Frequency Response (cont.)

◊ The transfer function $H(f)$ is a characteristic property of a LTI system. It is a complex quantity: $H(f) = |H(f)| \exp[j\beta(f)]$

◊ $|H(f)|$: *amplitude response*

◊ $\beta(f)$: *phase* or *phase response*

◊ If the impulse response $h(t)$ is real-valued, the transfer function $H(f)$ exhibits conjugate symmetry:

$$|H(f)| = |H(-f)| \qquad \beta(f) = -\beta(-f)$$

◇ Frequency Response (cont.)

  ◇ Illustrating the definition of *system bandwidth*

Low-pass system of bandwidth B

Band-pass system of bandwidth $2B$
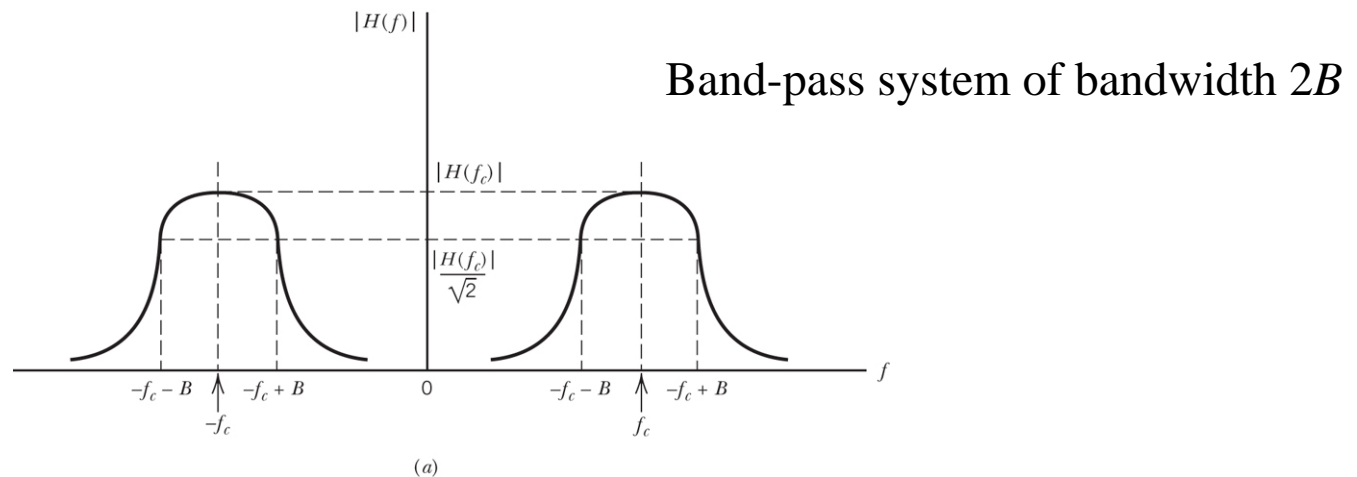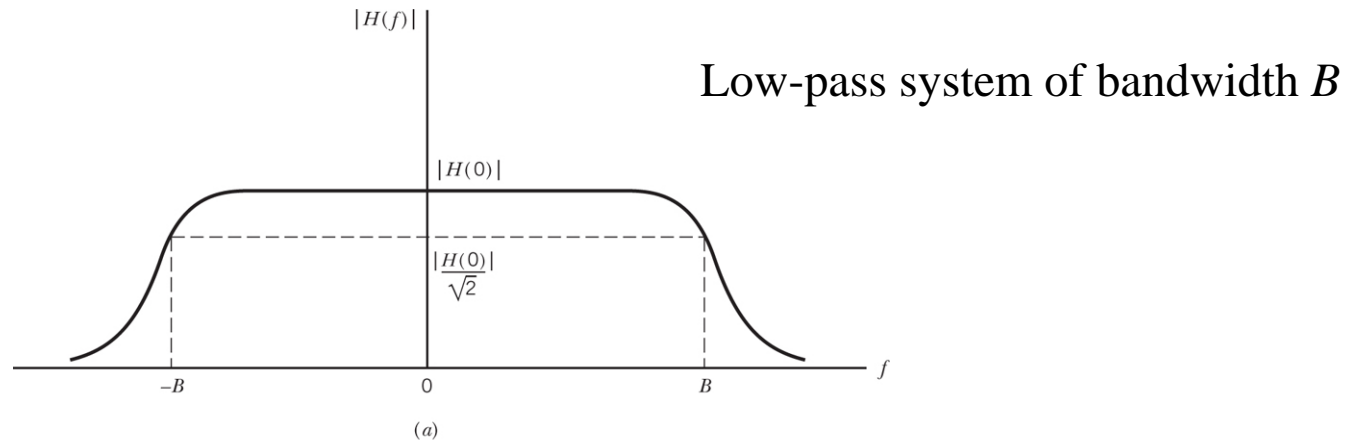
◇ Paley-Wiener Criterion

◇ A necessary and sufficient condition for a function $\alpha(f)$ to be the gain of a <u>causal</u> filter is the convergence of the integral

$$\int_{-\infty}^{\infty} \frac{|\alpha(f)|}{1+f^2} df < \infty$$

this condition is known as the *Paley-Wiener criterion*.

◇ We may associate with this gain a suitable phase $\beta(f)$, such that the resulting filter has a causal impulse response that is zero for negative time.

◇ <u>The Paley-Wiener criterion is the frequency-domain equivalent of the causality requirement.</u>

◇ A realizable gain characteristic may have infinite attenuation for a discrete set of frequencies, but it cannot have infinite attenuation over a band of frequencies.

Chapter 2.8 Filters

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2.8 Filters

◇ A filter is a <u>frequency-selective device</u> that is used to limit the spectrum of a signal to some specified band of frequencies.

◇ Frequency response is characterized by a *passband* and a *stopband*.

◇ The frequencies inside the passband are transmitted with little or no distortion, whereas those in the stopband are rejected.

◇ There are low-pass, high-pass, band-pass, and band-stop filters.

low-pass $|H(f)|$

f

0

band-pass $|H(f)|$

f

0

high-pass $|H(f)|$

f

0

band-stop $|H(f)|$

f

0
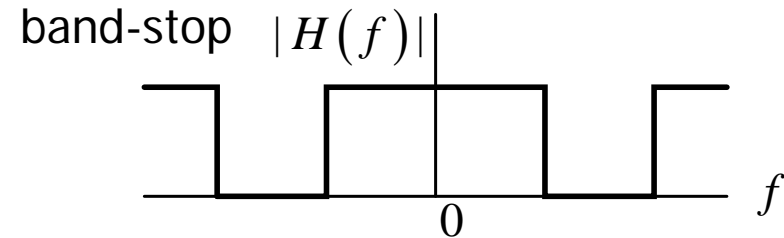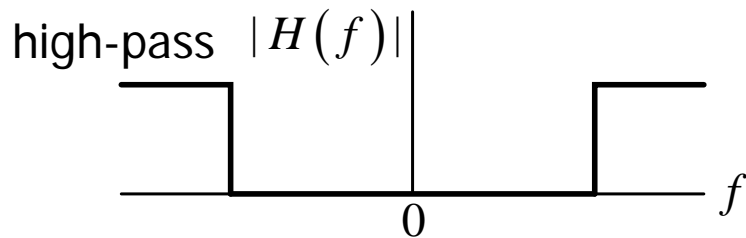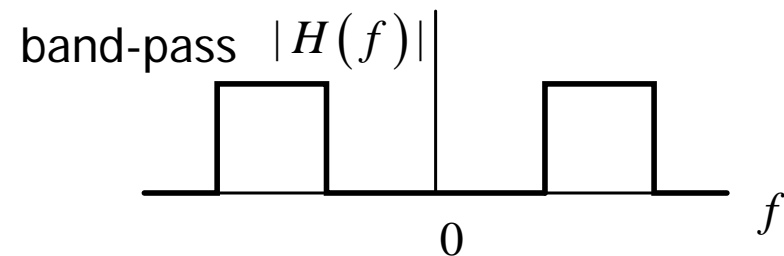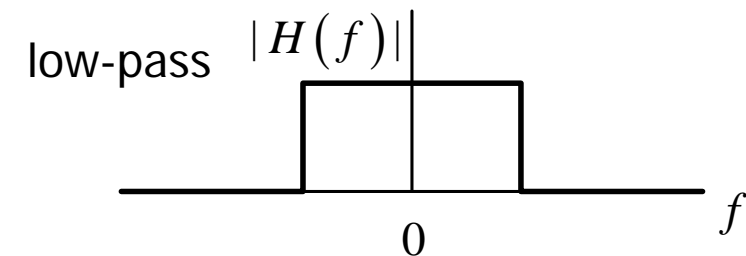
2.8 Filters

◇ Time response of the ideal low-pass filter

$$x(t) \longrightarrow \boxed{\text{LPF}} \longrightarrow y(t) = x(t - t_0)$$
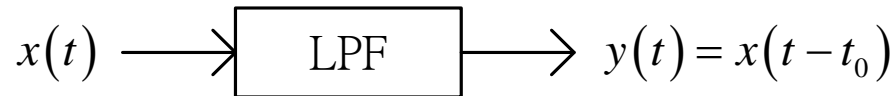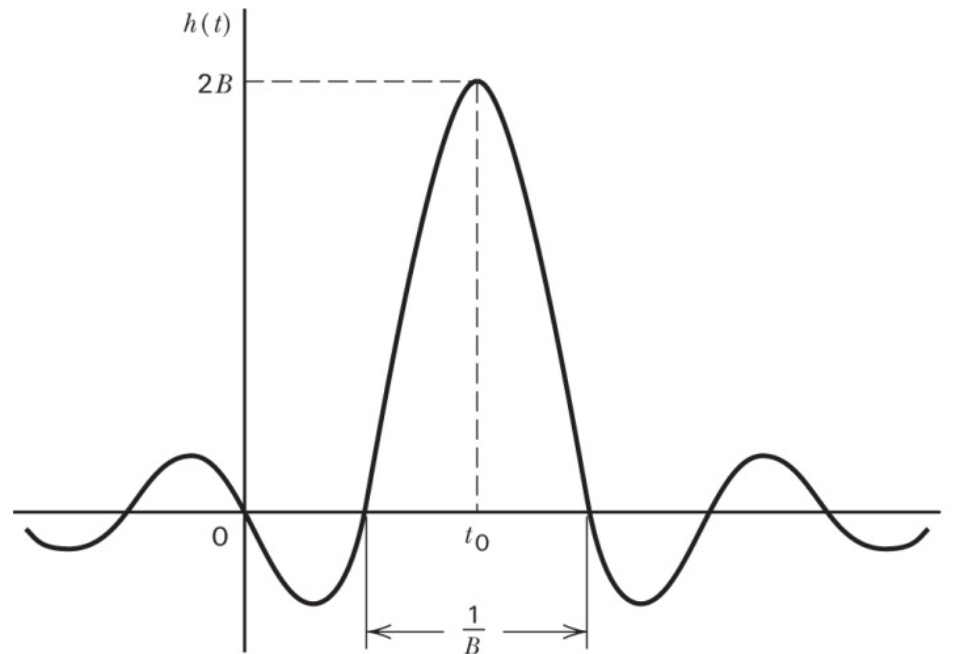
   ◇ The transfer function of an <u>ideal low-pass filter</u> is defined by:

$$H(f) = \begin{cases} \exp(-j2\pi f t_0), & -B \leq f \leq B \\ 0, & |f| > B \end{cases}$$

   ◇ The ideal low-pass filter is <u>noncausal</u> because it violates the Paley-Wiener criterion.

   ◇ This can be confirmed by examining the impulse response $h(t)$

$$h(t) = \int_{-B}^{B} \exp\left[j2\pi f (t - t_0) \right] df$$

$$= \frac{\sin\left[2\pi B(t - t_0) \right]}{\pi(t - t_0)} = 2B\,\text{sinc}\left[2B(t - t_0) \right] \qquad (2.118)$$

◇ There is some response from the filter before the time t=0, so confirming that <u>the ideal low-pass filter is noncausal</u>.

◇ However, we can make the delay t_0 large enough such that
$$\left|\text{sinc}\left[2B(t-t_0)\right]\right| \ll 1 \quad \text{for } t < 0$$

◇ By so doing, we are able to build a causal filter that closely approximates an ideal low-pass filter.

◇ [Example 2.13] Pulse response of ideal low-pass filter

◇ Consider a rectangular pulse $x(t)$ of unit amplitude and duration T, which is applied to an ideal low-pass filter of bandwidth B. The problem is to determine the response $y(t)$ of the filter.

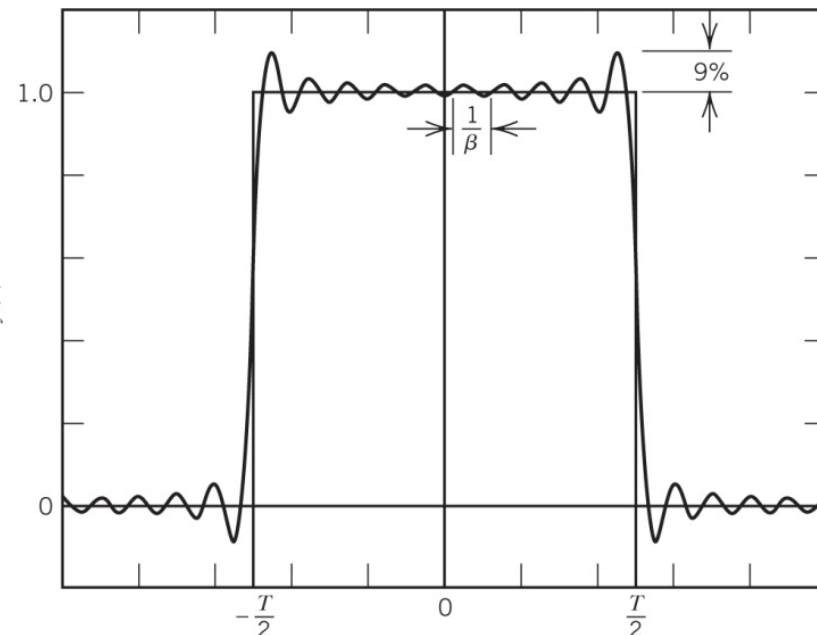◇ Using Eq. (2.118), and setting $t_0=0$ for simplification

$$h(t) = 2B\,\mathrm{sinc}(2Bt)$$

the resulting filter response

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$= 2B\int_{-T/2}^{T/2} \frac{\sin\left[2\pi B(t-\tau)\right]}{2\pi B(t-\tau)}\,d\tau$$

(no closed form)

Gibbs phenomenon

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2.8 Filters

◇ Design of Filters

  ◇ Design of filters is usually carried out in the <u>frequency domain</u>. There are two basic steps:

    ◇ The approximation of a prescribed frequency response(i.e. amplitude response, phase response, or both) by a realizable transfer function.

    ◇ The realization of the approximating transfer function by a physical device.

  ◇ For an approximating transfer function $H(f)$ to be physically realizable, it must represent a <u>stable</u> system.

  ◇ Stability is defined here on the basis of the bounded input bounded output criterion described in Eq. (2.100).

  ◇ In the following, we specify the corresponding condition for stability in terms of the transfer function.

  ◇ The traditional approach is to replace $j2\pi f$ with s.

2.8 Filters

◇ Design of Filters

◇ Ordinarily, the approximating transfer function $H'(s)$ is a rational function, which may be expressed in a factored form as:

$$H'(s) = H(f)\Big|_{j2\pi f = s}$$

$$= K\frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

where K is scaling factor; z_1, z_2, …, z_m are the *zeros* of the transfer function; p_1, p_2,…, p_n are its *poles*.

◇ For low-pass and band-pass filters: $m<n$.

◇ If the system is causal, all the poles of the transfer function $H'(s)$ should be inside the left half of the s-plane, i.e. $\text{Re}[p_i]<0$.
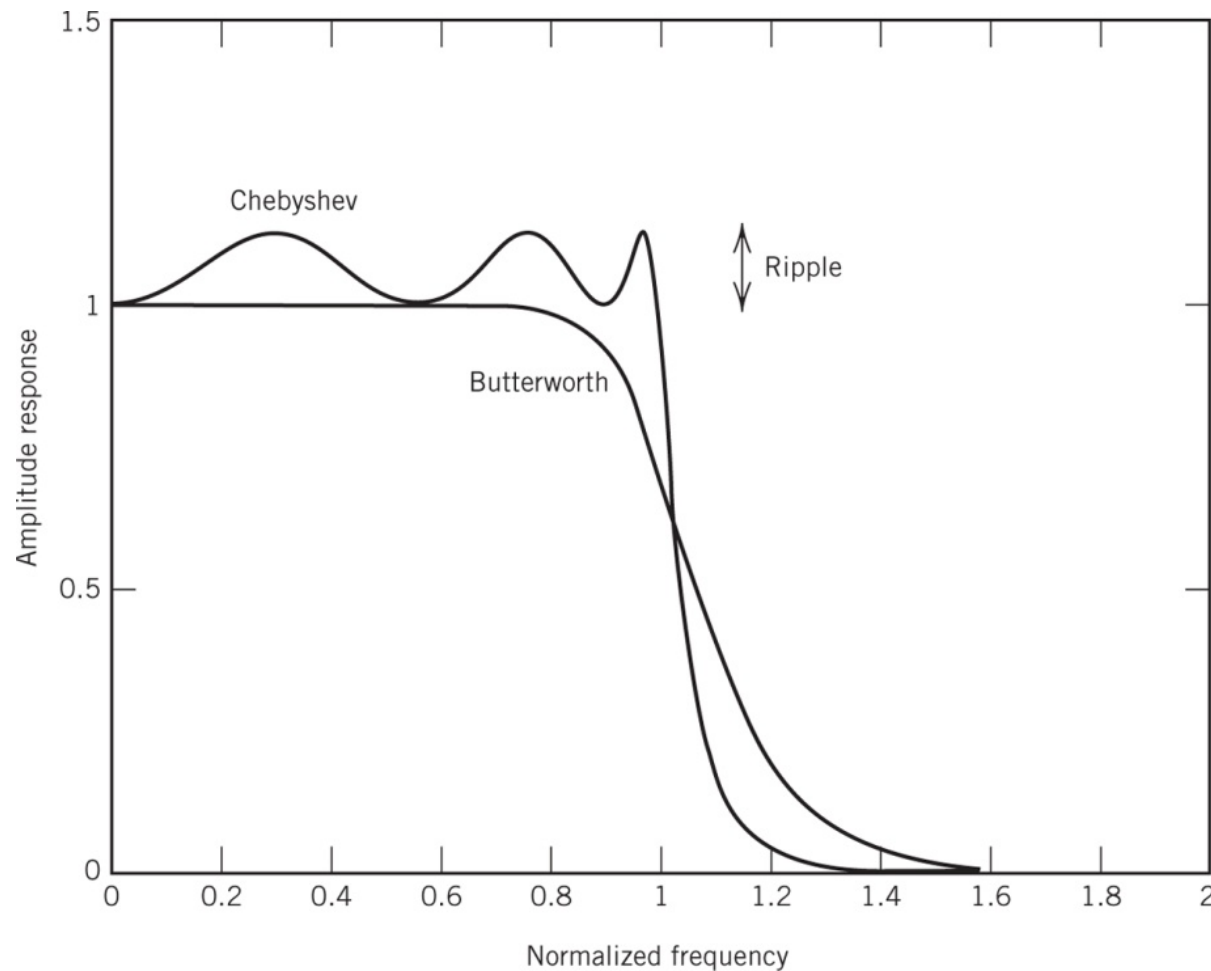
2.8 Filters

◊ Different Types of Filters

 ◊ Two popular families of low-pass filters: *Butterworth filters* and *Chebyshev filters*. All their zero are at $s= \infty$ and the poles are confined to the left half of the *s*-plane.

 ◊ Butterworth filter

  ◊ The poles of the transfer function lie on a circle with origin as the center and $2\pi B$ as the radius, where *B* is the 3-dB bandwidth of the filter.

  ◊ Is said to have a maximally flat passband response.

 ◊ Chebyshev filter

  ◊ The poles lie on an ellipse.

  ◊ Provide faster roll-off than Butterworth filter by allowing ripple in the frequency response.

  ◊ Type 1 filters have ripple only in the passband.

  ◊ Type 2 filters have ripple only in the stopband and are seldom used.

2.8 Filters

◇ Comparison of the amplitude response of 6th order Butterworth low-pass filter with that of 6th order Chebyshev filter.
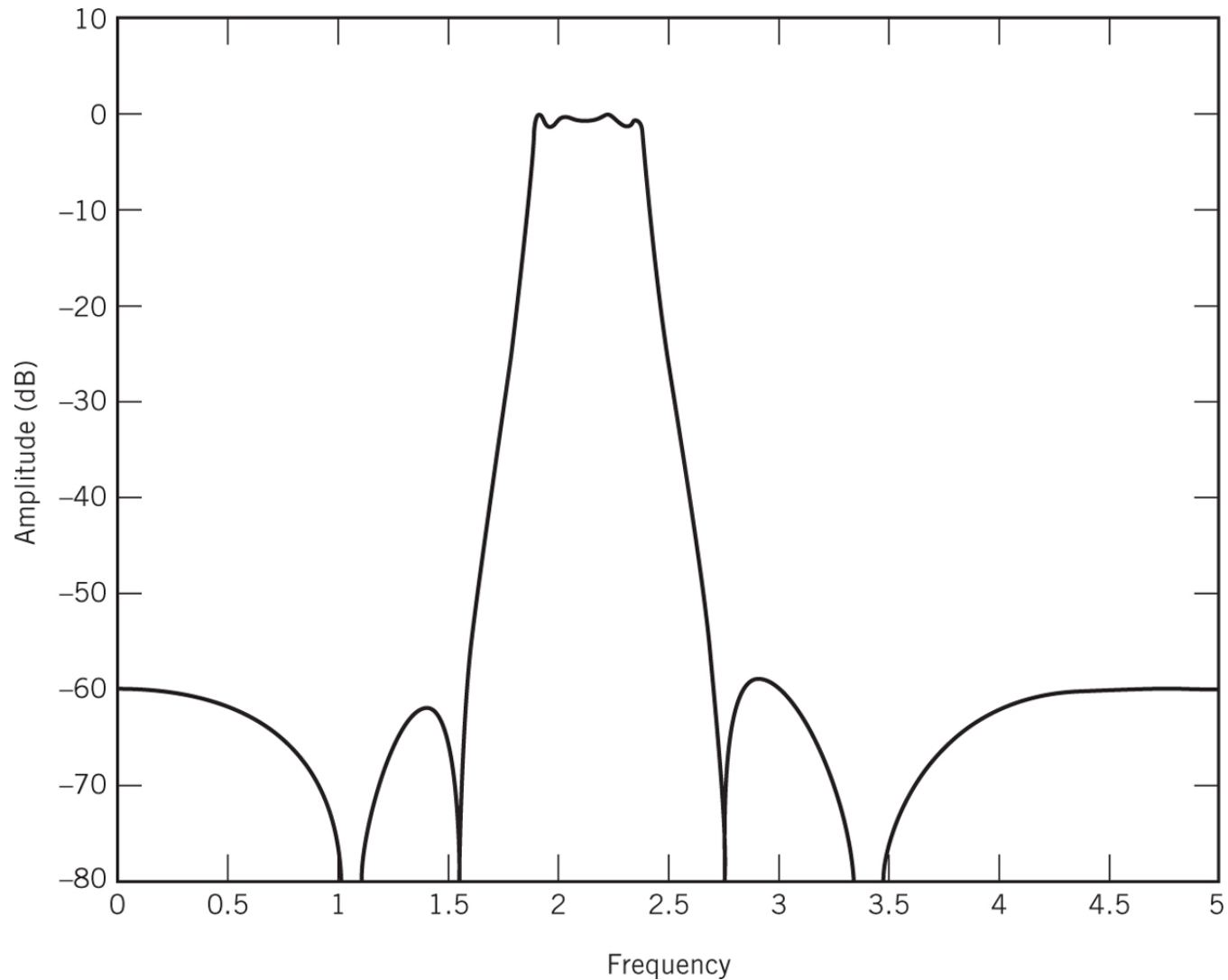
2.8 Filters

- A common alternative to both the Butterworth and Chebyshev filters is the *elliptic filter*, which has ripple in both the passband and the stopband.

- Elliptic filter provide even faster roll-off for a given number of poles but at the expense of ripple in both the passband and stopband.

- Butterworth filters are the simplest and elliptic filters are the more complicated to design in mathematical terms.

- The *finite-duration impulse response* (FIR) filter is often used in digital signal processing.

- The FIR filter is the equivalent of the tapped delay-line filter described in the previous section.

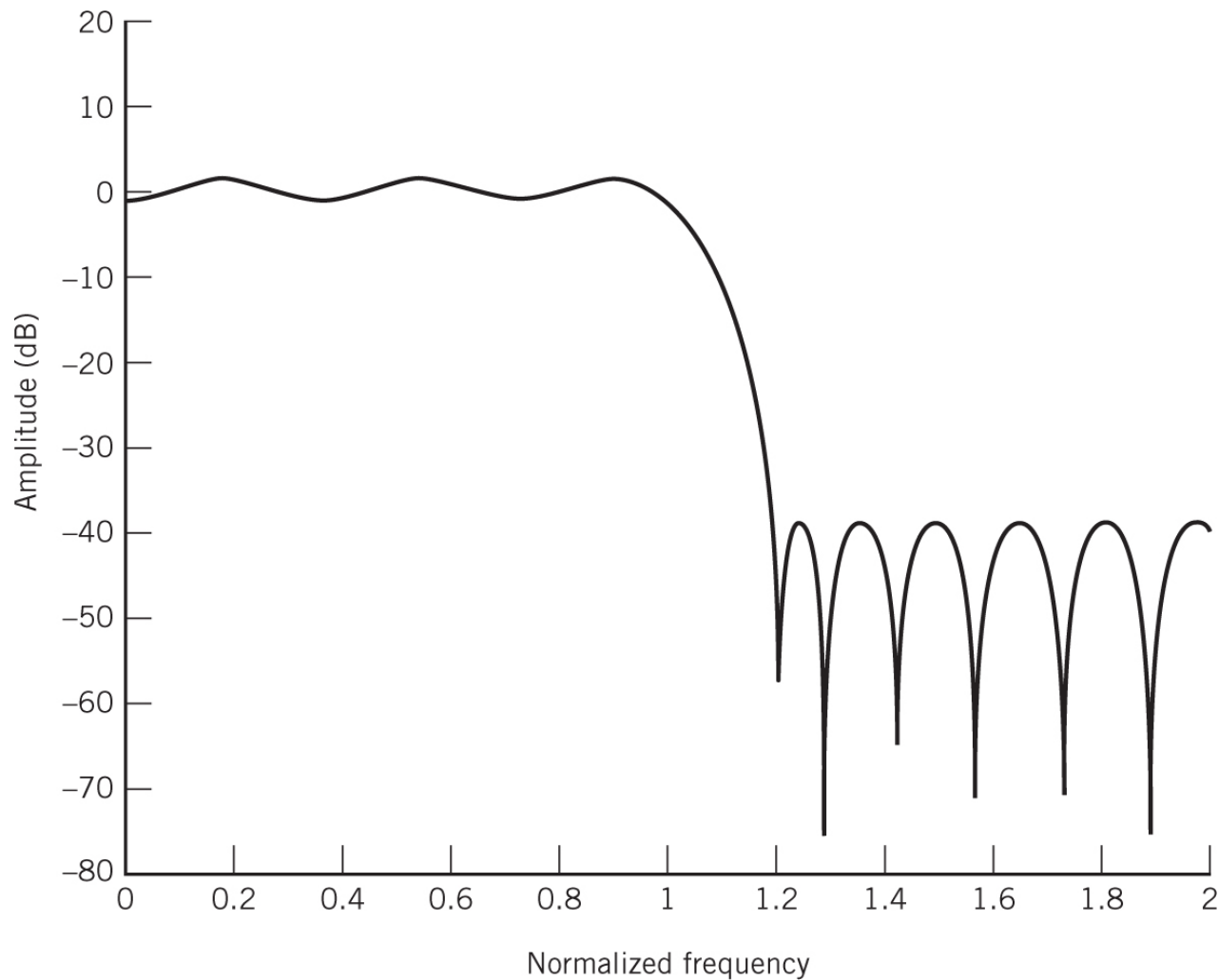- The FIR filter has only zeros; it is thus inherently stable.

2.8 Filters

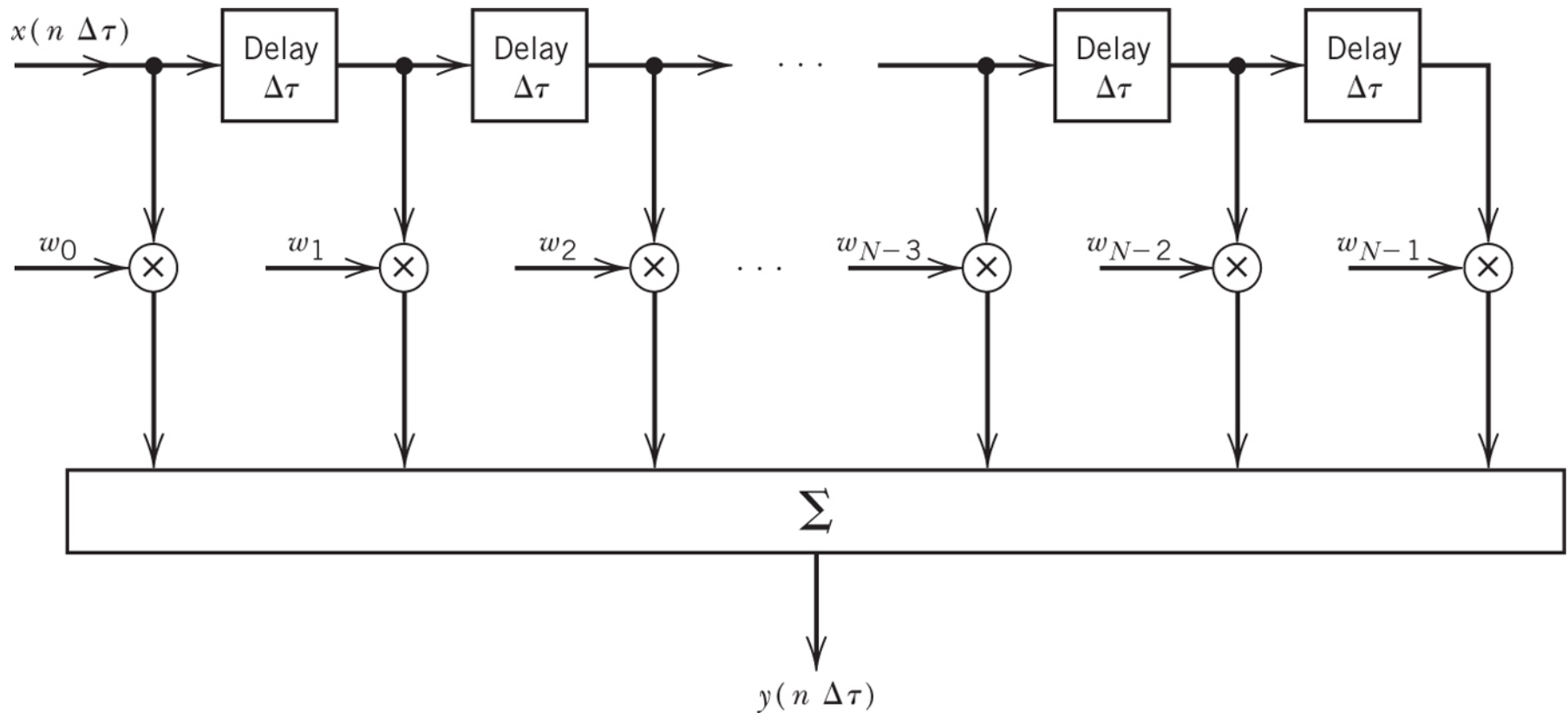◊ Amplitude response of 8th order elliptic bandpass filter.

◇ Amplitude response of 29-tap FIR low-pass filter.

2.8 Filters

◇ Tapped-delay-line Filter (FIR Filter)

Chapter 2.9
Low-Pass and Band-Pass Signals

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2.9 Low-Pass and Band-Pass Signals

◇ Communication using low-pass signals is referred to as *baseband communication*.

◇ In some transmission media, there is insufficient spectrum at baseband (e.g., radio waves) or the properties of media are not conductive to conducting signal at baseband (e.g., optical fibers). In these cases, we employ *band-pass communications*.
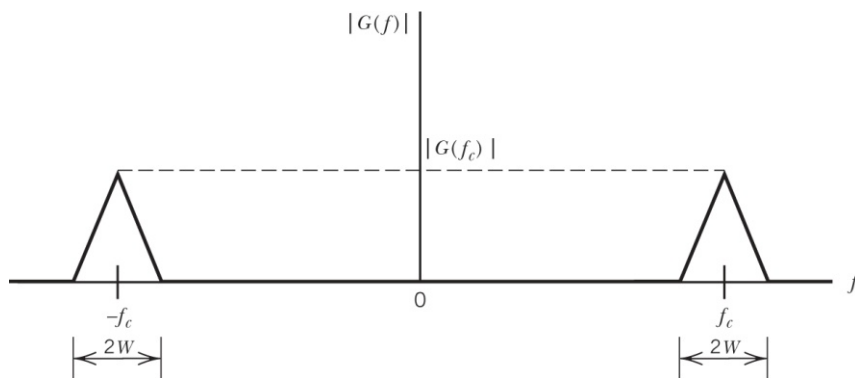
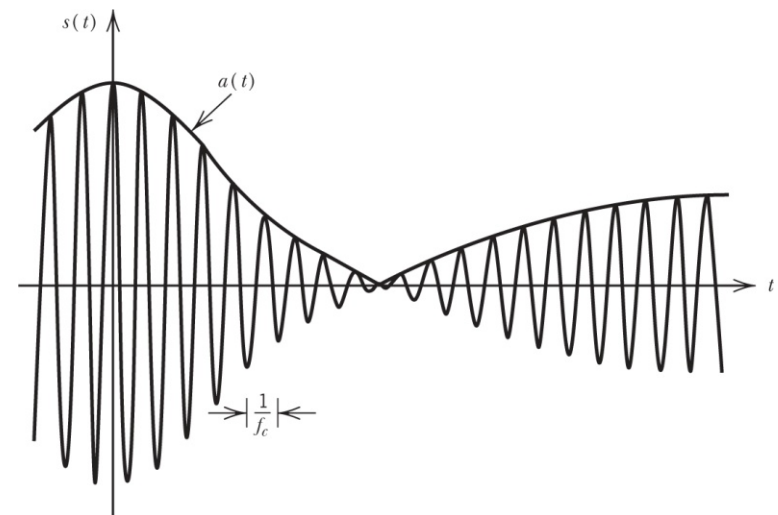Illustration of spectrum of band-pass signal.   Illustration of time-domain band-pass signal.

2.9 Low-Pass and Band-Pass Signals

◊ *Narrow-band signal*: the bandwidth $2W$ is small compared to the carrier frequency f_0.

◊ A real-valued band-pass signal $g(t)$ with non-zero spectrum $G(f)$ in the vicinity of f_c may be expressed in the form:

$$g(t) = a(t)\cos\left[2\pi f_c t + \phi(t)\right]$$

  ◊ a(t): envelope (non-negative)

  ◊ $\phi(t)$: phase

◊ Using the relationship cos(A+B)=cos(A)cos(B)-sin(A)sin(B)

$$g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \qquad (2.123)$$

$$g_I(t) = a(t)\cos\phi(t) \quad \text{and} \quad g_Q(t) = a(t)\sin\phi(t)$$

$$a(t) = \sqrt{g_I^2(t) + g_Q^2(t)}$$

$$\phi(t) = \tan^{-1}\left(\frac{g_Q(t)}{g_I(t)}\right)$$

in-phase of $g(t)$

quadrature of $g(t)$

2.9 Low-Pass and Band-Pass Signals

◇ Complex Baseband Representation

◇ Eq. (2.123) may be written as

$$g(t) = \text{Re}\left[\tilde{g}(t)\exp(j2\pi f_c t)\right] \qquad (2.126)$$

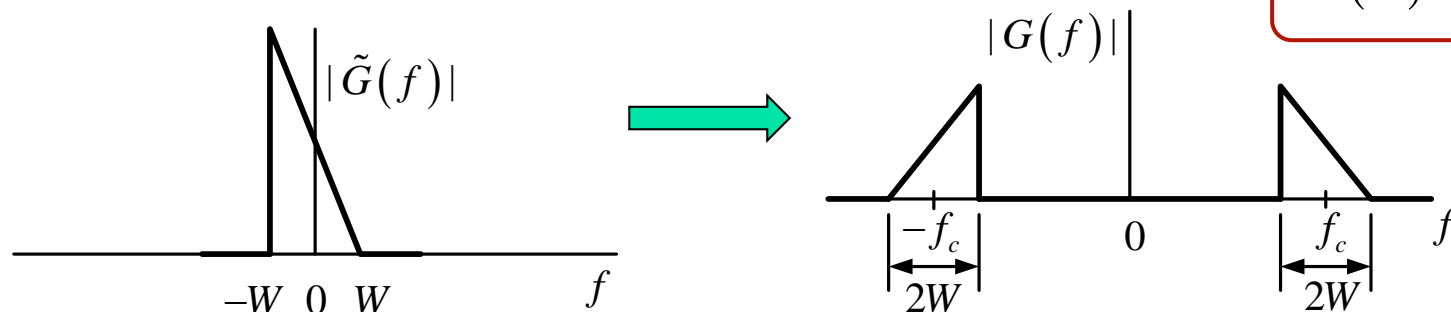where we define $\tilde{g}(t) = g_I(t) + jg_Q(t)$

◇ The $g_I(t)$ and $g_Q(t)$ are real, we refer to $\tilde{g}(t)$ as the *complex envelope* of the band-pass signal.

$$g(t) = \frac{1}{2}\left[\tilde{g}(t)\exp(j2\pi f_c t) + \tilde{g}^*(t)\exp(-j2\pi f_c t)\right]$$

$$\xrightarrow{F} \quad G(f) = \frac{1}{2}\left[\tilde{G}(f - f_c) + \tilde{G}^*(-f - f_c)\right]$$

$$g^*(t) \rightleftharpoons G^*(-f)$$

$$\text{Re}(A) = \frac{1}{2}\left(A + A^*\right)$$

◇ [Example 2.14] PF pulse (continued) – read by yourself

   ◇ To determine the complex envelope of the RF pulse

$$g(t) = A \, \text{rect}\left(\frac{t}{T}\right) \cos(2\pi f_c t)$$

   ◇ Assume $f_c T \gg 1$, so that $g(t)$ is narrow-band

$$g(t) = \text{Re}\left[A \, \text{rect}\left(\frac{t}{T}\right) \exp(j 2\pi f_c t) \right]$$

the complex envelope is

$$\tilde{g}(t) = A \, \text{rect}\left(\frac{t}{T}\right)$$

and the envelope equals

$$a(t) = \left| \tilde{g}(t) \right| = A \, \text{rect}\left(\frac{t}{T}\right)$$

Chapter 2.10
Band-Pass Systems

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◇ Summaries of low-pass systems:

  ◇ $x(t)$ represents the message signal, $y(t)$ is the received or output signal, and $h(t)$ is the impulse response of the channel or filter.

  ◇ $X(f)=F[x(t)]$, $H(f)=F[h(t)]$, $Y(f)=F[y(t)]$.

  ◇ Time domain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

  ◇ Frequency domain $\quad Y(f) = H(f)X(f)$

  ◇ These equations are valid for linear systems.

◇ Band-pass system

  ◇ Time domain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

  ◇ Frequency domain $\quad Y(f) = H(f)X(f)$

2.10 Band-Pass Systems

◇ Band-pass systems $\quad\boxed{g(t) = \text{Re}\left[\tilde{g}(t)\exp(j2\pi f_c t)\right]} \quad (2.126)$

    ◇ When $h(t)$ is the impulse response of a bandpass filter, by analogy with g(t) of Eq. 2.126, it may be represented as

$$h(t) = \text{Re}\left[\tilde{h}(t)\exp(j2\pi f_c t)\right]$$

where $\tilde{h}(t)$ is the *complex impulse response* of the bandpass filter.

    ◇ This response and its Fourier transform may be expressed as

$$h(t) = \text{Re}\left[\tilde{h}(t)\exp(j2\pi f_c t)\right]$$

$$\boxed{\text{Re}(A) = \frac{1}{2}\left(A + A^*\right)}$$

$$= \frac{1}{2}\left[\tilde{h}(t)\exp(j2\pi f_c t) + \tilde{h}^*(t)\exp(-j2\pi f_c t)\right]$$

$$H(f) = \frac{1}{2}\left[\tilde{H}(f - f_c) + \tilde{H}^*(-f - f_c)\right] \quad \text{(analogous to 2.129)}$$

Positive frequency portions of $H(f)$ Negative frequency portions of $H(f)$

2.10 Band-Pass Systems

◇ Since $\tilde{H}(f)$ is low-pass limited to $|f| < B$, we can obtain

$$\tilde{H}(f - f_c) = \begin{cases} 2H(f), & f > 0 \\ 0, & f < 0 \end{cases}$$

$$\tilde{H}(-f - f_c) = \begin{cases} 0, & f > 0 \\ 2H^*(f), & f < 0 \end{cases}$$

◇ This low-pass filter response is the frequency-domain equivalent of the complex impulse response of the filter.

◇ The output $y(t)$ is also a band-pass signal:

$$y(t) = \mathrm{Re}\left[\tilde{y}(t) \exp\left(j2\pi f_c t \right) \right]$$

where $\tilde{y}(t)$ is the complex envelope of $y(t)$.

$$Y(f) = H(f)X(f)$$

$$= \frac{1}{2}\left[\tilde{H}(f - f_c) + \tilde{H}^*(-f - f_c)\right] \times \frac{1}{2}\left[\tilde{X}(f - f_c) + \tilde{X}^*(-f - f_c)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\tilde{H}(f - f_c)\tilde{X}(f - f_c) + \frac{1}{2}\tilde{H}^*(-f - f_c)\tilde{X}^*(-f - f_c)\right]$$

$$= \frac{1}{2}\left[\tilde{Y}(f - f_c) + \tilde{Y}^*(-f - f_c)\right]$$

where $\tilde{Y}(f) = \dfrac{1}{2}\tilde{H}(f)\tilde{X}(f)$ (2.139)

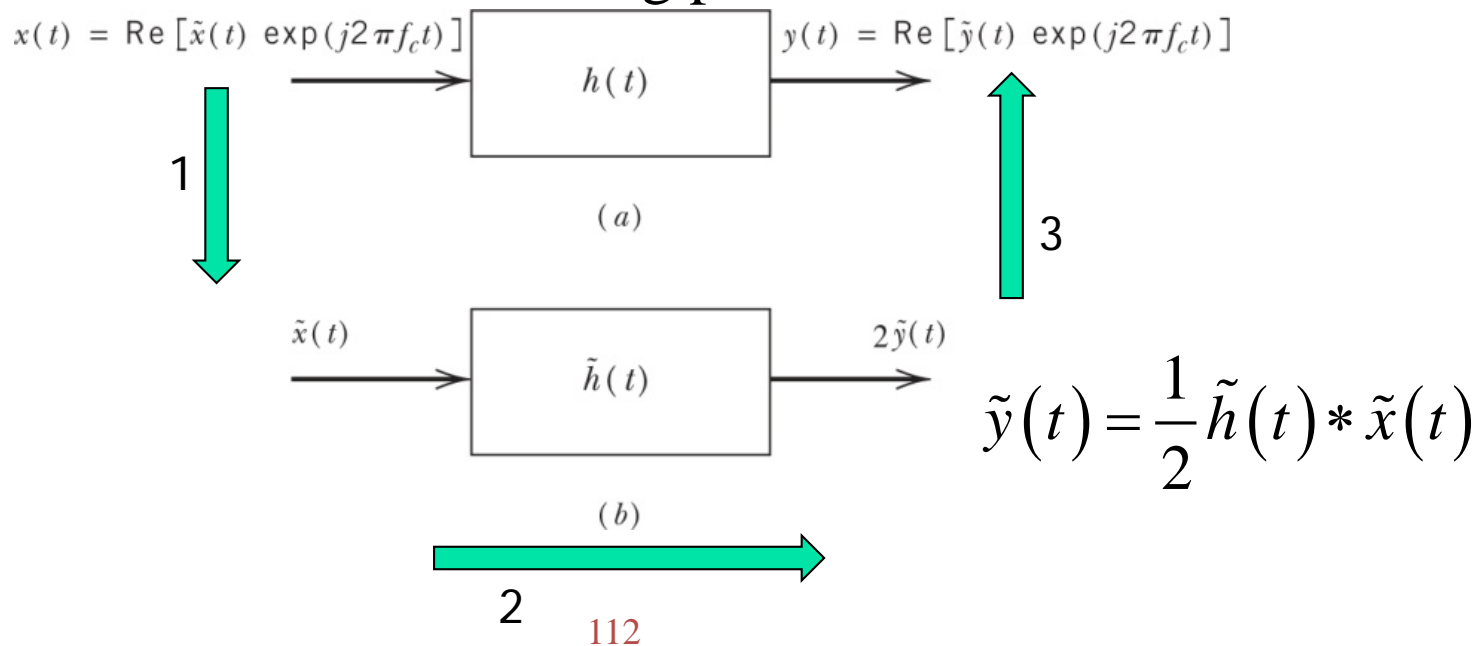$$\tilde{y}(t) = \frac{1}{2}\tilde{h}(t) * \tilde{x}(t) \qquad (2.140)$$

$$\boxed{\begin{aligned} &\tilde{H}(f - f_c)\tilde{X}^*(-f - f_c) \\ &= \tilde{H}^*(-f - f_c)\tilde{X}(f - f_c) \\ &= 0 \end{aligned}}$$

◇ The complex envelope of the band-pass output is the convolution of the complex envelope of the filter and the input, scaled by the factor ½.

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2.10 Band-Pass Systems

◇ The analysis of a band-pass system is complicated due to the multiplying factors $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$.

◇ The significance of Eq. (2.140) is that, we need only concern the low-pass functions, $\tilde{x}(t), \tilde{h}(t),$ and $\tilde{y}(t)$.

◇ In other words, the analysis of a band-pass system is replaced by an equivalent but much simpler low-pass analysis that completely retains the essence of the filtering process.
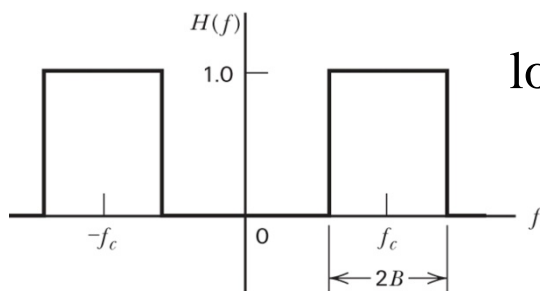
$$x(t) = \mathrm{Re}\left[\tilde{x}(t)\,\exp(j2\pi f_c t)\right] \qquad h(t) \qquad y(t) = \mathrm{Re}\left[\tilde{y}(t)\,\exp(j2\pi f_c t)\right]$$

$$(a)$$

$$\tilde{x}(t) \qquad \tilde{h}(t) \qquad 2\tilde{y}(t)$$

$$(b)$$

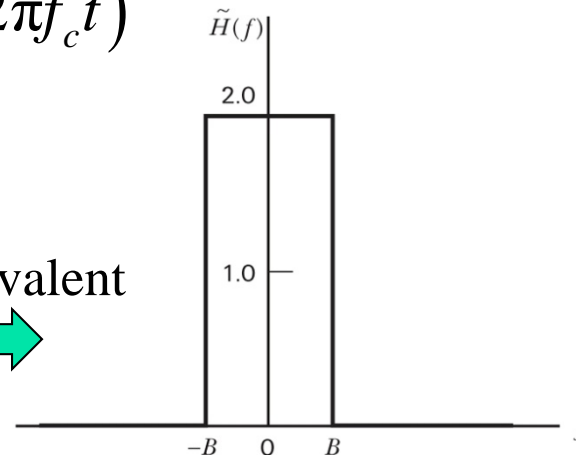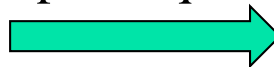$$\tilde{y}(t) = \frac{1}{2}\tilde{h}(t) * \tilde{x}(t)$$

2.10 Band-Pass Systems

◊ [Example 2.15] Response of an ideal band-pass filter to a pulsed RF wave

  ◊ Target: compute the response of an ideal band-pass filter $H(f)$ to an RF pulse of duration T and carrier frequency f_c ($f_c\,T >> 1$)

$$x(t) = A\,\text{rect}\left(\frac{t}{T}\right)\cos\left(2\pi f_c t\right)$$

low-pass equivalent

$$\tilde{H}(f) = \begin{cases} 2, & -B < f < B \\ 0, & |f| > B \end{cases} \xrightarrow{\ F\ } \tilde{h}(t) = 4B\,\text{sinc}(2Bt)$$
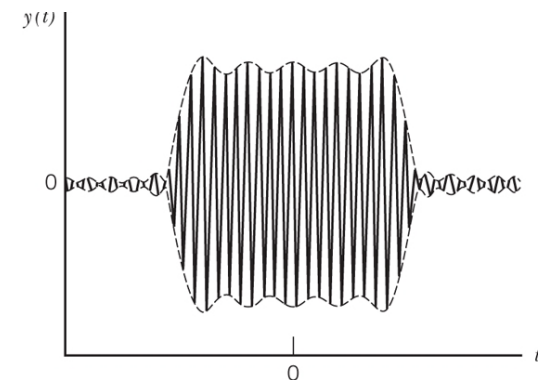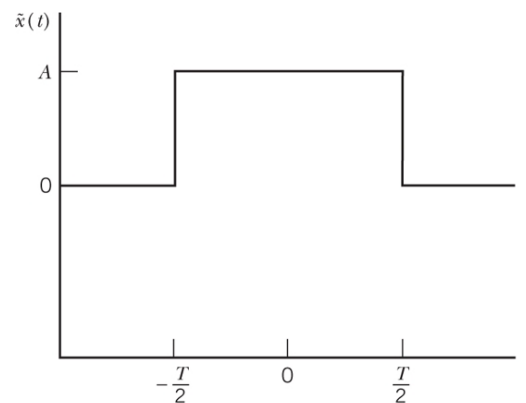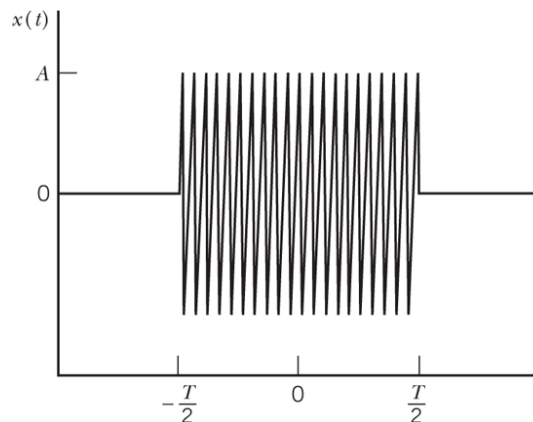
$$x(t) = A \operatorname{rect}\left(\frac{t}{T}\right) \cos\left(2\pi f_c t\right)$$

low-pass equivalent
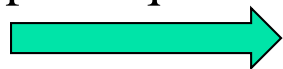
$$\tilde{x}(t) = A \operatorname{rect}\left(\frac{t}{T}\right)$$

$$\Rightarrow \tilde{y}(t) = \frac{1}{2}\tilde{h}(t) * \tilde{x}(t) \qquad \text{(no closed form)}$$

Chapter 2.11
Phase and Group Delay

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2.11 Phase and Group Delay

◇ Whenever a signal is transmitted through a dispersive (frequency-selective) device such as a filter or communication channel, some delay is introduced into the output signal in relation to the input signal.

◇ In an ideal filter, the phase response varies *linearly* with frequency inside the passband of the filter, in which case the filter introduces a constant delay.

◇ *Question: what if the phase response of the filter is nonlinear?*

◇ Signal Models: assume that a steady sinusoidal signal at frequency f_c is transmitted through a dispersive channel that has a total phase-shift of $\beta(f_c)$.

  ◇ *Phase delay* of the channel: $\beta(f_c)/2\pi f_c$ [sec] is the time taken by the received signal phasor to sweep out this phase lag.

◇ Phase delay is not necessarily the true signal delay.

◇ The true signal delay is represented by the *envelope* or *group delay*.

◇ Assume that the dispersive channel is described by the transfer function:

$$H(f) = K \exp\left[j\beta(f) \right]$$

where K is a constant and the phase $\beta(f)$ is a nonlinear function of frequency.

◇ The input signal $x(t)$ consists of a narrow-band signal:

$$x(t) = m(t)\cos\left(2\pi f_c t\right)$$

where $m(t)$ is a low-pass (information-bearing) signal with its spectrum limited to the frequency interval $|f| \leqq W$. Assume $f_c \gg W$.

◇ By using the Taylor series about the point $f=f_c$ and retaining only the first two terms:

$$\beta(f) \simeq \beta(f_c) + (f - f_c)\frac{\partial\beta(f)}{\partial f}\bigg|_{f=f_c}$$

Taylor series at $f = f_c$

$$\sum_{n=0}^{\infty}\frac{\beta^{(n)}(f_c)}{n!}(f - f_c)^n$$

◇ Define *phase delay*: $\tau_p = -\dfrac{\beta(f_c)}{2\pi f_c}$

◇ Define *group delay*: $\tau_g = -\dfrac{1}{2\pi}\dfrac{\partial \beta(f)}{\partial f}\Big|_{f=f_c}$

$$\beta(f) \simeq \beta(f_c) + (f - f_c)\dfrac{\partial \beta(f)}{\partial f}\Big|_{f=f_c} \Rightarrow \beta(f) \simeq -2\pi f_c \tau_p - 2\pi(f - f_c)\tau_g$$

◇ The transfer function of the channel takes the form:

$$H(f) \simeq K \exp\Big[-j2\pi f_c \tau_p - j2\pi(f - f_c)\tau_g\Big]$$

◇ Equivalent low-pass filter

$$\tilde{H}(f) = \begin{cases} 2H(f - f_c), & f - f_c > 0 \\ 0, & f - f_c < 0 \end{cases}$$

$$(2.134)$$

$$\tilde{H}(f) \simeq 2K \exp\left(-j2\pi f_c \tau_p - j2\pi f \tau_g\right)$$

◇ Low-pass complex envelope and its Fourier transform:

$$\tilde{x}(t) = m(t), \ \tilde{X}(f) = M(f) \triangleq F\big[m(t)\big]$$

2.11 Phase and Group Delay

◇ The Fourier transform of the complex envelope of the received signal:

$$\tilde{Y}(f) = \frac{1}{2}\tilde{H}(f)\tilde{X}(f) \qquad (2.139)$$

$$\simeq K\exp\left(-j2\pi f_c \tau_p\right)\exp\left(-j2\pi f \tau_g\right)M(f)$$

◇ The term $\exp(-j2\pi f\tau_g)M(f)$ represents the Fourier transform of the delayed signal $m(t-\tau_g)$.

◇ Complex envelope of the received signal:

$$\tilde{y}(t) \simeq K\exp\left(-j2\pi f_c \tau_p\right)m\left(t-\tau_g\right)$$

◇ Received signal:

$$y(t) = \mathrm{Re}\left[\tilde{y}(t)\exp\left(j2\pi f_c t\right)\right]$$

$$= Km\left(t-\tau_g\right)\cos\left[2\pi f_c\left(t-\tau_p\right)\right]$$

2.11 Phase and Group Delay

◇ The sinusoidal carrier wave $\cos(2\pi f_c t)$ is delay by τ_p seconds, hence τ_p represents the *phase delay*. Sometimes, τ_p is also referred to as the *carrier delay*.

◇ The envelope $m(t)$ is delayed by τ_g seconds; hence, τ_g represents the *envelope* or *group delay*.

◇ τ_g is related to the slope of the phase $\beta(f)$, measured at $f=f_c$.

◇ When the phase response $\beta(f)$ varies linearly with frequency, the signal is delayed but undistorted.

◇ When this linear condition is violated, we get *group delay distortion*.
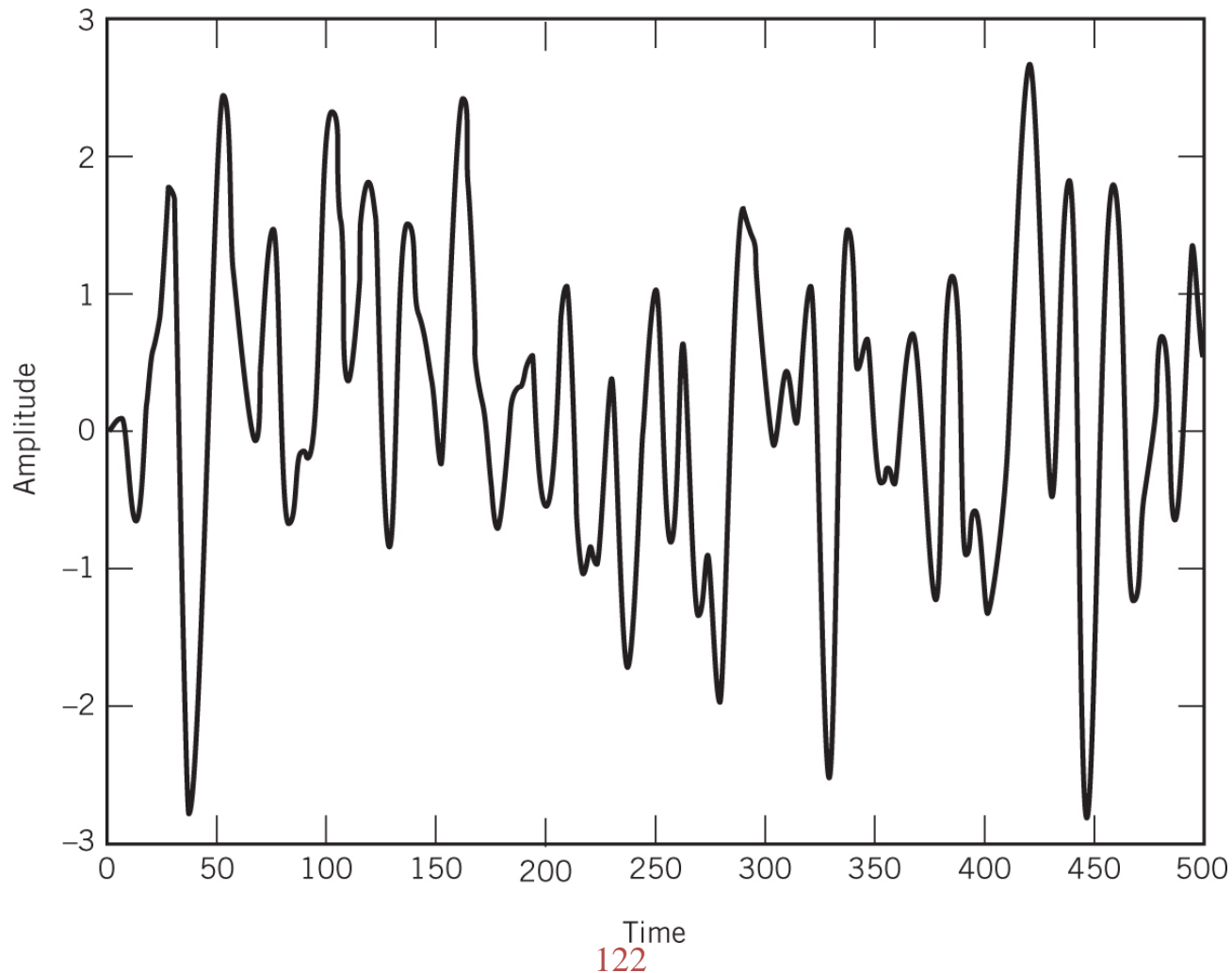
Chapter 2.12
Sources of Information

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2.12 Sources of Information

◊ An example of waveform that represents an analog source of information.

◇ Some source are digital in the sense that the information can be naturally represented as a sequence of zeros and ones.

◇ The digital waveform can be represented as:

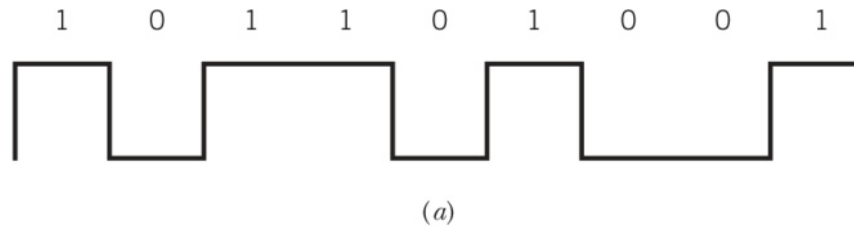$$g(t) = \sum_{k=0}^{K} b_k \, p(t - kT)$$

1   0   1   1   0   1   0   0   1

(a)

Figure 2.38 (a) rectangular pulse shape

$Ap(t\text{-}0T)$ $Ap(t\text{-}2T)$ $Ap(t\text{-}3T)$ · · ·
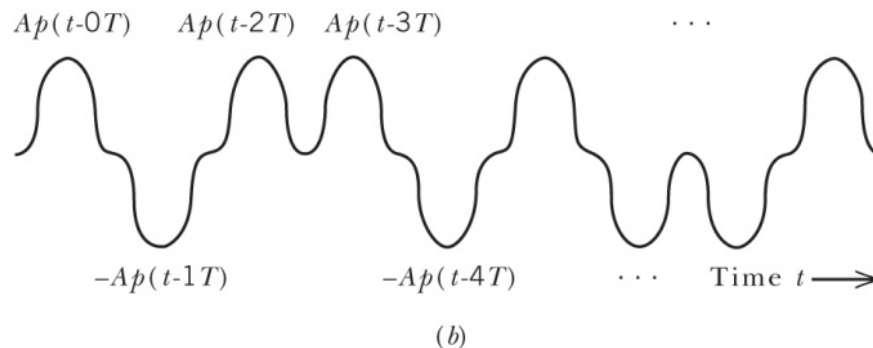
$-Ap(t\text{-}1T)$ $-Ap(t\text{-}4T)$ · · ·   Time $t \longrightarrow$

(b)

Figure 2.38 (b) non-rectangular pulse shape