HW 1.

\# 2.1

(a) $g(t) = A\cos(2\pi f_c t)\,\mathrm{rect}\left(\frac{t}{T}\right)$

$f_c = \frac{1}{T_s} = \frac{1}{2T}$

$\Rightarrow G(f) = A \cdot \mathcal{F}\{\cos(2\pi f_c t)\} \circledast \mathcal{F}\{\mathrm{rect}(\frac{t}{T})\}$

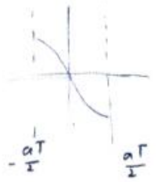
$= A \cdot \{\frac{1}{2}\delta(f-f_c) + \frac{1}{2}\delta(f+f_c)\} \cdot T\,\mathrm{sinc}(Tf)$

$= \frac{AT}{2}\{\mathrm{sinc}(T(f+f_c)) + \mathrm{sinc}(T(f-f_c))\}$ *

(b) $g_2(t) = g(t - \frac{T}{2})$

$= G(f)e^{-j2\pi f\frac{T}{2}} = G(f)e^{-j\pi fT}$ *

(c) $g_3(t) = g(t - \frac{aT}{2})$



$= G(f)e^{-j2\pi f\cdot\frac{aT}{2}}$

$= G(f)e^{-j\pi f aT}$ *

(d) $g_4(t) = -g(t + \frac{T}{2})$

$\Rightarrow -G(f)e^{j2\pi f\frac{T}{2}}$

$= -G(f)e^{j\pi fT}$ *

(e) $g_5(t) = g_2(t) + g_4(t)$

$G(f) = G(f)e^{-j\pi fT} - G(f)e^{j\pi fT}$

$= G(f)\,(\cos(\pi fT) - j\sin(\pi fT) - (\cos(\pi fT) + j\sin(\pi fT)))$

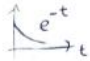
$= G(f)\,(-2j\sin(\pi fT))$ *

\# 2.2

$g(t) = e^{-t}\sin(2\pi f_c t)\,u(t)$

$= e^{-t}u(t) \cdot \sin(2\pi f_c t)$

$\mathcal{F} \Rightarrow \mathcal{F}\{e^{-t}u(t)\} \circledast \{\frac{1}{2j}\delta(f-f_c) - \frac{1}{2j}\delta(f+f_c)\}$



$= \int_0^\infty e^{-t}e^{-j2\pi ft}dt$

$= \int_0^\infty e^{-t(1+j2\pi f)}dt$

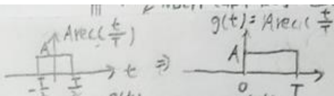
$= \frac{1}{-1-j2\pi f}[0-1]$

$= \frac{1}{1+j2\pi f}$

$= \frac{1}{2j}\left[\frac{1}{1+j2\pi(f-f_c)} - \frac{1}{1+j2\pi(f+f_c)}\right]$ *

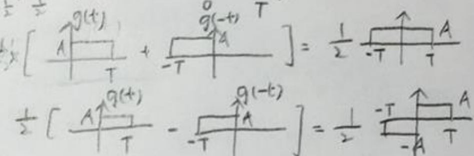
3.

$g(t) = A\,\mathrm{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$



(1) $g_e(t) = \frac{1}{2}[g(t) + g(-t)] \Rightarrow \frac{1}{2}[\ldots] = \frac{1}{2}\ldots = \frac{A}{2}\mathrm{rect}(\frac{t}{2T})$ *

$g_o(t) = \frac{1}{2}[g(t) - g(-t)] \Rightarrow \frac{1}{2}[\ldots] = \frac{1}{2}\ldots = \frac{1}{2}A\,\mathrm{rect}(\frac{t}{T} - \frac{1}{2}) + \frac{1}{2}A\,\mathrm{rect}(\frac{t}{T} + \frac{1}{2})$ *



(2) 由图，

$G_e(f) = \mathcal{F}\{\frac{A}{2}\mathrm{rect}(\frac{t}{2T})\} = \frac{A}{2} \cdot 2T\,\mathrm{sinc}(2Tf)$

$= AT\,\mathrm{sinc}(2fT)$ *

$G_o(f) = \mathcal{F}\{\frac{A}{2}[\mathrm{rect}(\frac{t-\frac{T}{2}}{T}) + \mathrm{rect}(\frac{-t-\frac{1}{2}T}{T})]$

$= \frac{AT}{2}[\mathrm{sinc}(Tf)e^{j2\pi\frac{T}{2}f} + \mathrm{sinc}(Tf)e^{-j2\pi\frac{T}{2}f}]$

cos j   +(cos j)

$= \frac{AT}{2}\mathrm{sinc}(Tf)[-2j\sin(\pi Tf)]$

$= \frac{AT}{j}\mathrm{sinc}(fT)\sin(\pi fT)$ *

2.4.

$$G(f) = \begin{cases} 1 \cdot e^{j(-\frac{\pi}{2})} = -j & , 0 < f < W \\ 1 \cdot e^{j(\frac{\pi}{2})} = j & , -W < f < 0 \end{cases}$$

$$g(t) = \int_0^W -j e^{j2\pi f t} df + \int_{-W}^0 j e^{j2\pi f t} df$$

$$= -j \frac{1}{j2\pi t}\left(e^{j2\pi f t}\Big|_{f=0}^{W}\right) + j \frac{1}{j2\pi t}\left(e^{j2\pi f t}\Big|_{-W}^{0}\right)$$

$$= \frac{-1}{2\pi t}\left(e^{j2\pi t W} - 1\right) + \frac{1}{2\pi t}\left(1 - e^{-j2\pi t W}\right)$$

$$= \frac{-1}{2\pi t} \underbrace{e^{j2\pi t W}}_{\cos + j\sin} + \frac{1}{2\pi t} + \frac{1}{2\pi t} - \frac{1}{2\pi t} \underbrace{e^{-j2\pi t W}}_{\cos - j\sin}$$

$$= -\frac{1}{2\pi t}\left[2\cos(2\pi t W)\right] + \frac{1}{\pi t} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1-\cos(2\pi t W)}{\pi t} \ \# \quad \leftarrow \quad \frac{2\sin^2(\pi t W)}{\pi t} = 2W \operatorname{sinc}(tW)\sin(\pi t W) \ \#$$

#2.5

(a) even 偶 $\Rightarrow$ $g(t)=g(-t)$, $\int_{-\infty}^{\infty} g(-t)e^{-j2\pi ft}dt$; $-t=\tau$; $\int_{-\infty}^{\infty} g(\tau)e^{j2\pi f\tau}d\tau$; $=G^*(f)$

$\Rightarrow$ $\downarrow$ F.T

$G(f)=G^*(f)$

$a+jb \quad a-jb$

$\Rightarrow G(f)$ is real #

odd $\Rightarrow$ $g(-t)=-g(t)$

$\downarrow$ F.T

$\Rightarrow$ $G^*(f)=-G(f)$

$a-jb=-(a+jb)$

$=-a-jb$

$\Rightarrow G(f)$ is image #

$g(t)\rightarrow G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}dt$

$g^*(t)\rightarrow G_1(f)=\int_{-\infty}^{\infty} g^*(t)e^{-j2\pi ft}dt$

$=\left(\int_{-\infty}^{\infty} g(t)e^{j2\pi ft}dt\right)^*$

$G^*(-f)$

(b) $X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

$\frac{d^n}{df^n}X(f)=\int_{-\infty}^{\infty} x(t)\frac{d^n}{df^n}e^{-j2\pi ft}dt$

$=\int_{-\infty}^{\infty} x(t)(-j2\pi t)^n e^{-j2\pi ft}dt$

$=(-j2\pi)^n\int_{-\infty}^{\infty} \underbrace{x(t)t^n} e^{-j2\pi ft}dt$

$=(-j2\pi)^n \mathcal{F}\{x(t)\cdot t^n\}$

$\Rightarrow \mathcal{F}\{x(t)\cdot t^n\} = \left(\frac{j}{2\pi}\right)^n \frac{d^nX(f)}{df^n}$

(c) $\Rightarrow \int_{-\infty}^{\infty} t^n x(t)e^{-j2\pi ft}dt$

如題 $f=0$ $=\left(\frac{j}{2\pi}\right)^n \frac{d^nX(0)}{df^n}$ #

(d)

$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\,dt$

$\int_{-\infty}^{\infty} g_1(t)\int_{-\infty}^{\infty} G_2^*(-f)e^{j2\pi ft}df\,dt$

$=\int_{-\infty}^{\infty} g_1(t)\int_{-\infty}^{\infty} G_2^*(f')e^{-j2\pi f't}df'\,dt$

$=\int_{-\infty}^{\infty} G_2^*(f)\,G_1(f)\,df$ #

(e) $g_1(t)g_2^*(t)$

$\xrightarrow{F.T.} \mathcal{F}\{g_1(t)\} * \mathcal{F}\{g_2^*(t)\}$

$= G_1(f) * G_2^*(-f)$

$=\int_{-\infty}^{\infty} G_1(\lambda)G_2^*(-(f-\lambda))\,d\lambda$

$=\int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda-f)\,d\lambda$ #

#2.6

(a) F.T. $g_1(t)*g_2(t) \xrightarrow{F.T.} G_1(f)\cdot G_2(f)$

$=G_2(f)\cdot G_1(f) \xrightarrow{F.T.^{-1}} g_2(t)*g_1(t)$ #

(b) F.T.

$g_1(t)*[g_2(t)*g_3(t)] \xrightarrow{F.T.} G_1(f)\cdot(G_2(f)\cdot G_3(f))$

$=[G_1(f)\cdot G_2(f)]\,G_3(f) \xrightarrow{F.T.^{-1}} (g_1(t)*g_2(t))*g_3(t)$

(c) F.T.

$g_1(t)*[g_2(t)+g_3(t)] \xrightarrow{F.T.} G_1(f)\cdot[G_2(f)+G_3(f)]$

$=G_1(f)G_2(f)+G_1(f)G_3(f) \xrightarrow{F.T.^{-1}} g_1(t)*g_2(t)+$

$g_1(t)*g_3(t)$ #

Prom 7 by $F\{\frac{d^n x(t)}{dt^n}\} = (j2\pi f)^n X(f)$

(1) $\frac{d}{dt}[g_1(t) * g_2(t)] \xrightarrow{F.T.} (j2\pi f)\ F\{g_1(t) * g_2(t)\}$

$= (j2\pi f)(G_1(f) \cdot G_2(f))$

$= [j2\pi f \cdot G_1(f)] \cdot G_2(f)$

$\downarrow F^{-1}.T.$

$= \left(\frac{d}{dt} g_1(t)\right) * g_2(t)$ #

(2) by $F\{\int_{-\infty}^{t} x(\tau)\,d\tau\} = \frac{1}{j2\pi f} X(f) + \cancel{\frac{1}{2} X(0)\,\delta(f)}$

$\Rightarrow \int_{-\infty}^{t} [g_1(\tau) * g_2(\tau)]\,d\tau \xrightarrow{F.T.} \frac{1}{j2\pi f}\{G_1(f)\,G_2(f)\}$

$= \{\frac{1}{j2\pi f} G_1(f)\} \cdot G_2(f)$

$\xrightarrow{F^{-1}.T.} \left(\int_{-\infty}^{t} g_1(\tau)\,d\tau\right) * g_2(t)$ #

Prom 8

(1) $g(t) = rect(\frac{t}{T})$

$G(f) = T\,sinc(Tf)$

$\Rightarrow g_{new}(t) = \frac{1}{T} rect(\frac{t}{T})$

$G_{new}(f) = sinc(Tf)$ → 單位化

$\Rightarrow \lim_{T\to 0} g_{new}(t) \to \delta(t)$

$\lim_{T\to 0} G(f) \to 1$

(2) $g(t) = 2W\,sinc(2Wt)$

$G(f) = rect(\frac{f}{2W})$

$\lim_{W\to 0} g(t) = \delta(t)$

$\lim_{W\to 0} G(f) = 1$

Prom 9 : $G(f) = \frac{1}{2} sgn(f) + \frac{1}{2}$

$\to g(t) = \frac{1}{2}\delta(t) + \frac{j}{2\pi t}$ #

by $sgn(t) = \frac{1}{j\pi f} \Rightarrow sgn(f) = -\frac{1}{j\pi t} = \frac{j}{\pi t}$

10. Given: $\int_{-\infty}^{\infty}|x(t)|^2 < \infty$ and $\int_{-\infty}^{\infty}|h(t)|\,dt < \infty$

by Rayleigh's Thm $\int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty}|X(f)|^2 df < \infty$

$\Rightarrow \quad y(t) = x(t) \circledast h(t)$

$Y(f) = H(f)\,X(f)$

$\Rightarrow$ by Rayleigh's Thm

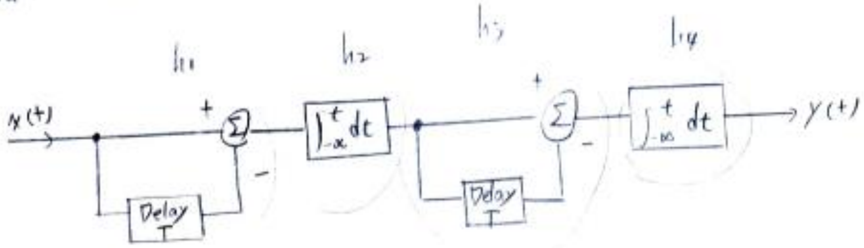
$$\int_{-\infty}^{\infty}|y(t)|^2 dt = \int_{-\infty}^{\infty}|Y(f)|^2 df = \int_{-\infty}^{\infty} Y(f)\,Y^*(f)\,df$$

$$= \int_{-\infty}^{\infty} H(f)X(f)\cdot H^*(f)X^*(f)\,df$$

$$= \int_{-\infty}^{\infty}|X(f)|^2|H(f)|^2 df \underset{\text{柯西不等式}}{\le} \overset{<\infty}{\int_{-\infty}^{\infty}|X(f)|^2 df} \cdot \overset{<\infty}{\int_{-\infty}^{\infty}|H(f)|^2 df}$$

$\therefore \int_{-\infty}^{\infty}|y(t)|^2 dt < \infty$ #

11.



$y(t) = x(t) \circledast h_1(t) \circledast h_2(t) \circledast h_3(t) \circledast h_4(t)$

$\Rightarrow Y(f) = X(f)\cdot H_1(f)\cdot H_2(f)\cdot H_3(f)\cdot H_4(f)$

$\Rightarrow$ by ① $h_1(t) = h_3(t) = \delta(t) - \delta(t-T)$

$\rightarrow H_1(f) = H_3(f) = 1 - e^{-j2\pi fT}$

② $h_2(t) = h_4(t) = \int_{-\infty}^{t} dt$

$\rightarrow H_2(f) = H_4(f) = \frac{1}{j2\pi f}$

$$\therefore H(f) = \frac{Y(f)}{X(f)} = H_1(f)\cdot H_2(f)\,H_3(f)\,H_4(f)$$

$$= (1-e^{-j2\pi fT})^2\cdot\left(\frac{1}{j2\pi f}\right)^2$$

$$= \frac{-1}{(2\pi f)^2}\left[\,1 - 2e^{-j2\pi fT} + e^{-j4\pi fT}\,\right]$$ #