2019-Chapter5 solution

Problem 1:

Given $f(x)=\frac{1}{\sqrt{2\pi\sigma_x^2}}\exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right)$ and $\exp(-\pi t^2)\Leftrightarrow\exp(-\pi f^2)$, then by applying the time-shifting and scaling properties:

$$
\begin{aligned}
F(f)&=\frac{1}{\sqrt{2\pi\sigma_x^2}}\left|\sqrt{2\pi\sigma_x^2}\right|\exp\left(-\pi\left(\sqrt{2\pi\sigma_x^2}\right)^2\pi f^2\right)\exp(j2\pi f\mu_x)\\
&=\exp\left(-\pi^2 2\sigma_x^2 f^2+j\mu_x 2\pi f\right)\text{ and let } v=2\pi f\\
&=\exp\left(jv\mu_x-\frac{1}{2}v^2\sigma_x^2\right)
\end{aligned}
$$

另解

$$
\begin{aligned}
\phi_X(v)&=\int_{-\infty}^{\infty}e^{ivx}f(x)dx\\
&=\int_{-\infty}^{\infty}e^{ivx}\frac{1}{\sqrt{2\pi\sigma_x^2}}e^{\left(\frac{-(x-\mu_x)^2}{2\sigma_x^2}\right)}dx\\
&=\frac{1}{\sqrt{2\pi\sigma_x^2}}\int_{-\infty}^{\infty}e^{\frac{-2\sigma_x^2}{-2\sigma_x^2}ivx}e^{\left(\frac{-1}{2\sigma_x^2}\right)\left(x^2-2x\mu_x+\mu_x^2\right)}dx\\
&=\frac{1}{\sqrt{2\pi\sigma_x^2}}\int_{-\infty}^{\infty}e^{\frac{1}{-2\sigma_x^2}\left(x^2-\left(2\mu_x+2\sigma_x^2iv\right)x\right)}e^{\frac{\mu_x^2}{-2\sigma_x^2}}dx\\
&=\frac{1}{\sqrt{2\pi\sigma_x^2}}\int_{-\infty}^{\infty}e^{\frac{1}{-2\sigma_x^2}\left(x^2-2\left(\mu_x+\sigma_x^2iv\right)x+\left(\mu_x+\sigma_x^2iv\right)^2\right)}e^{\frac{\mu_x^2}{-2\sigma_x^2}}e^{\frac{-\left(\mu_x+\sigma_x^2iv\right)^2}{-2\sigma_x^2}}dx\\
&=\frac{1}{\sqrt{2\pi\sigma_x^2}}\int_{-\infty}^{\infty}e^{\frac{1}{-2\sigma_x^2}\left(x-\left(\mu_x+\sigma_x^2iv\right)\right)^2}dx\; e^{\frac{\mu_x^2-\mu_x^2-2\mu_x\sigma_x^2iv+\sigma_x^4v^2}{-2\sigma_x^2}}\\
&=e^{\frac{-2\mu_x\sigma_x^2iv+\sigma_x^4v^2}{-2\sigma_x^2}}\\
&=e^{\mu_x iv-\frac{1}{2}\sigma_x^2v^2}
\end{aligned}
$$

Problem 2:

For a complex random process $Z(t)$, $R_Z(\tau) = E\left[Z^*(t)Z(t+\tau)\right]$, then

i. The mean square of a complex process is given by

$$R_Z(0) = E\left[Z^*(t)Z(t)\right] = E\left[|Z(t)|^2\right]$$

ii. We show $R_Z(\tau)$ has conjugate symmetry by the following

$$R_Z(-\tau) = E\left[Z^*(t)Z(t-\tau)\right] = E\left[Z^*(\mathrm{s}+\tau)Z(\mathrm{s})\right] = E\left[Z^*(\mathrm{s})Z(\mathrm{s}+\tau)\right]^* = R_Z^*(\tau)$$

where we have used the change of variable $s = t - \tau$.

iii. Taking an approach similar to that of $|R_X(\tau)| \le R_X(0)$

$$\begin{aligned}
0 &\le E\left[\left|Z(t) \pm Z(t+\tau)\right|^2\right] \\
&= E\left[\left(Z(t) \pm Z(t+\tau)\right)\left(Z^*(t) \pm Z^*(t+\tau)\right)\right] \\
&= E\left[Z(t)Z^*(t) \pm Z(t)Z^*(t+\tau) \pm Z^*(t)Z(t+\tau) + Z(t+\tau)Z^*(t+\tau)\right] \\
&= E\left[|Z(t)|^2\right] \pm E\left[Z(t)Z^*(t+\tau)\right] \pm E\left[Z^*(t)Z(t+\tau)\right] + E\left[|Z(t+\tau)|^2\right] \\
&= 2E\left[|Z(t)|^2\right] \pm 2\,\mathrm{Re}\left\{E\left[Z^*(t)Z(t+\tau)\right]\right\} \\
&= 2R_Z(0) \pm 2\,\mathrm{Re}\left\{R_z(\tau)\right\}
\end{aligned}$$

Thus $\left|\mathrm{Re}\left\{R_Z(\tau)\right\}\right| \le R_Z(0)$

Problem 3:

(a) $R_{XY}(\tau) = E\left[X(t+\tau)Y(t)\right]$, replacing $t+\tau$ with t':

$$R_{XY}(\tau) = E\left[X(t')Y(t'-\tau)\right] = R_{YX}(-\tau)$$

(b)

$$\begin{aligned}
E\left[\left(X(t+\tau) \pm Y(t)\right)^2\right] &= E\left[X^2(t+\tau)\right] \pm 2E\left[X(t+\tau)Y(t)\right] + E\left[Y^2(t)\right] \\
&= R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \ge 0
\end{aligned}$$

Thus, $\frac{1}{2}\left[R_X(0) + R_Y(0)\right] \ge |R_{XY}(\tau)|$

Problem 4:

(a) The filter output is $Y(t) = \int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau = \frac{1}{T}\int_0^T X(t-\tau) d\tau$

Put $t - \tau = u$. Then, the sample value of $Y(t)$ at $t = T$ equals $Y = \frac{1}{T}\int_0^T X(u) du$

The mean of Y is therefore $E[Y] = E\left[\frac{1}{T}\int_0^T X(u) du\right] = \frac{1}{T}\int_0^T E[X(u)] du = 0$

The variance of Y is

$$\sigma_Y^2 = E[Y^2] - \{E[Y]\}^2 = R_Y(0)$$

$$= \int_{-\infty}^{\infty} S_Y(f) df = \int_{-\infty}^{\infty} S_X(f) |H(f)|^2 df$$

But

$$H(f) = \int_{-\infty}^{\infty} h(t) \exp(-j2\pi ft) dt$$

$$= \frac{1}{T}\int_0^T \exp(-j2\pi ft) dt$$

$$= \frac{1}{T} \left. \frac{\exp(-j2\pi ft)}{-j2\pi f} \right|_0^T$$

$$= \frac{1}{j2\pi fT}\left[1 - \exp(-j2\pi fT)\right] = \sin c(fT) \exp(-j\pi fT)$$

Therefore, $\sigma_Y^2 = \int_{-\infty}^{\infty} S_X(f) \sin c^2(fT) df$

(b) Since the filter input is Gaussian, it follows that Y is also Gaussian. Hence, the probability density function of Y is

$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y}} \exp\left(\frac{-y^2}{2\sigma_Y^2}\right)$, where σ_Y^2 is defined above.