Communication Systems

2018–Ch4 exercises solution

Problem 1:

$$s(t) = A_c \cos(\theta(t))$$
$$\theta(t) = 2\pi f_c t + k_p m(t)$$

Let $\beta = 0.3$ for $m(t) = \cos(2\pi f_m t)$.

$$\therefore s(t) = A_c \cos(2\pi f_c t + \beta m(t))$$
$$= A_c[\cos(2\pi f_c t)\cos(\beta\cos(2\pi f_m t)) - \sin(2\pi f_c t)\sin(\beta\cos(2\pi f_m t))]$$

for small β:

$$\cos(\beta\cos(2\pi f_m t)) \approx 1$$
$$\sin(\beta\cos(2\pi f_m t)) \approx \beta\cos(2\pi f_m t)$$

$$\therefore s(t) = A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\cos(2\pi f_m t)$$
$$= A_c \cos(2\pi f_c t) - \beta \frac{A_c}{2}[\sin(2\pi(f_c + f_m)t) + \sin(2\pi(f_c - f_m)t)]$$

Problem 2:

The frequency deviation is

$$\Delta f = k_f A_m = 25\times 10^3 \times 20 = 5\times 10^5\,\text{Hz}$$

The corresponding value of the modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{5\times 10^5}{10^5} = 5$$

The transmission bandwidth of the FM wave, using Carson's rule, is therefore

$$B_T = 2 f_m (1+\beta) = 2\times 100(1+5) = 1200\,\text{kHz} = 1.2\,\text{MHz}$$

Problem 3:

The filter input is

$$v_1(t) = g(t)\cdot s(t)$$
$$= g(t)\cos(2\pi f_c t - \pi k t^2)$$

The complex envelope of $v_1(t)$ is

$$\tilde{v}_1(t) = g(t)\exp(-j\pi k t^2)$$

The impulse response $h(t)$ of the filter is defined in terms of the complex impulse response $\tilde{h}(t)$ as follows

$$h(t) = \mathrm{Re}[\tilde{h}(t)\exp(j2\pi f_c t)]$$

With

$$h(t) = \cos(2\pi f_c t + \pi k t^2),$$

we have

$$\tilde{h}(t) = \exp(j\pi k t^2)$$

The complex envelope of the filter output is therefore

$$\begin{aligned}
\tilde{v}_0(t) &= \frac{1}{2}\tilde{h}(t)\tilde{v}_i(t) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} g(\tau)\exp(-j\pi k t^2)\exp[j\pi k(t-\tau)^2)]d\tau \\
&= \frac{1}{2}\exp(j\pi k t^2)\int_{-\infty}^{\infty} g(\tau)\exp(-j\pi k t\tau)d\tau \\
&= \frac{1}{2}\exp(j\pi k t^2)G(kt)
\end{aligned}$$

Hence,

$$\tilde{v}_0(t) = \frac{1}{2}\left|G(kt)\right|$$

This shows that the envelope of the filter output is, expect for scale factor of $1/2$, equal to the magnitude of the Fourier transform of the input signal $g(t)$, with kt playing the role of frequency f.

Problem 4:

$v_2 = a v_1^2$
$s(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t))$
$\quad = A_c \cos(2\pi f_c t + \beta m(t))$

$$\begin{aligned} v_2 &= a \cdot s^2(t) \\ &= a \cdot A_c^2 \cos^2(2\pi f_c t + \beta m(t)) \\ &= \frac{A_c'}{2} \cdot (\cos(4\pi f_c t + 2\beta m(t)) + 1) \\ &= \frac{A_c'}{2} \cdot (\cos(2\pi(2 f_c)t + 2\beta m(t)) + 1) \end{aligned}$$

The square-law device produces a new FM signal centred as $2 f_c$ and with a frequency deviation of 2β. This doubles the frequency deviation.

Problem 5:
The envelope detector input is

$$\begin{aligned} v(t) &= s(t) - s(t-T) \\ &= A_c \cos[2\pi f_c t + \phi(t)] - A_c \cos[2\pi f_c (t-T) + \phi(t-T)] \\ &= -2A_c \sin\left[\frac{2\pi f_c (t-T) + \phi(t) + \phi(t-T)}{2}\right] \sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] \qquad (1) \end{aligned}$$

where

$\phi(t) = \beta \sin(2\pi f_m t)$

The phase difference $\phi(t) - \phi(t-T)$ is

$$\begin{aligned} \phi(t) - \phi(t-T) &= \beta \sin(2\pi f_m t) - \beta \sin[2\pi f_m (t-T)] \\ &= \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t)\cos(2\pi f_m T) + \cos(2\pi f_m t)\sin(2\pi f_m T)] \\ &\approx \beta[2\pi f_m t - 2\pi f_m t + 2\pi f_m T \cdot \cos(2\pi f_m t)] \\ &= 2\pi \Delta f T \cdot \cos(2\pi f_m t) \end{aligned}$$

where

$\Delta f = \beta f_m$.

Therefore, noting that $2\pi f_c T = \pi/2$, we may write

$$\sin\left[\frac{2\pi f_c(t-T)+\phi(t)-\phi(t-T)}{2}\right]\approx \sin[\pi f_c T+\pi\Delta fT\cdot\cos(2\pi f_m t)]$$
$$=\sin\left[\frac{\pi}{4}+\pi\Delta fT\cdot\cos(2\pi f_m t)\right]$$
$$=\frac{\sqrt{2}}{2}\cos[\pi\Delta fT\cdot\cos(2\pi f_m t)]+\frac{\sqrt{2}}{2}\sin[\pi\Delta fT\cdot\cos(2\pi f_m t)]$$
$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\pi\Delta fT\cdot\cos(2\pi f_m t)$$

where we have made use of the fact that $\pi\Delta fT \ll 1$. We may therefore rewrite Eq. (1) as

$$v(t)\approx -\sqrt{2}A_c[1+\pi\Delta fT\cdot\cos(2\pi f_m t)]\cdot\sin\left[\pi f_c(2t-T)+\frac{\phi(t)+\phi(t-T)}{2}\right]$$

Accordingly, the envelope detector output is

$$a(t)\approx\sqrt{2}A_c[1+\pi\Delta fT\cdot\cos(2\pi f_m t)]$$

which, except for a basis term, is proportional to the modulating wave.