Chapter 4
Optimum Receiver for AWGN Channels
4.1 Waveform and Vector Channel Models
4.2 Waveform and Vector AWGN Channels
4.3 Optimal Detection and Error Probability for Band-Limited Signaling
4.4 Optimal Detection and Error Probability for Power-Limited Signaling
4.5 Optimal Detection in Presence of Uncertainty: Non-Coherent Detection
4.6 A Comparison of Digital Signaling Methods
4.10 Performance Analysis for Wireline and Radio Communication Systems
Chapter 4.1
Waveform and Vector Channel Models

Wireless Information Transmission System Lab.
Institute of Communications Engineering
National Sun Yat-sen University
In this chapter, we study the effect of noise on the reliability of the modulation systems studied in Chapter 3.

We assume that the transmitter sends digital information by use of \( M \) signals waveforms \( \{s_m(t)\}_{1,2,\cdots,M} \). Each waveform is transmitted within the symbol interval of duration \( T \), i.e. \( 0 \leq t \leq T \).

The additive white Gaussian noise (AWGN) channel model:

\[
r(t) = s_m(t) + n(t)
\]

- \( s_m(t) \): transmitted signal
- \( n(t) \): sample function of AWGN process with PSD: \( \Phi_{nn}(f) = N_0/2 \) (W/Hz)
Based on the observed signal $r(t)$, the receiver makes the decision about which message $m$, $1 \leq m \leq M$ was transmitted

- **Optimum decision**: minimize error probability $P_e = P[\hat{m} \neq m]$

Any orthonormal basis $\{\phi_j(t), 1 \leq j \leq N\}$ can be used for expansion of a zero-mean white Gaussian process (Problem 2.8-1)

- The resulting coefficients are *i.i.d* zero-mean Gaussian random variables with variance $N_0/2$
- $\{\phi_j(t), 1 \leq j \leq N\}$ can be used for expansion of noise $n(t)$

Using $\{\phi_j(t), 1 \leq j \leq N\}$, $r(t) = s_m(t) + n(t)$ has the vector form

$$r = s_m + n$$

- All vectors are $N$-dimensional
- Components in $n$ are *i.i.d* zero-mean Gaussian with variance $N_0/2$
It is convenient to subdivide the receiver into two parts—the *signal demodulator* and the *detector*.

The function of the *signal demodulator* is to convert the received waveform $r(t)$ into an $N$-dimensional vector $\mathbf{r}=[r_1 \ r_2 \ \cdots \ r_N]$ where $N$ is the dimension of the transmitted signal waveform.

The function of the *detector* is to decide which of the $M$ possible signal waveforms was transmitted based on the vector $\mathbf{r}$. 
Two realizations of the signal demodulator are described in the next two sections:

- One is based on the use of signal correlators.
- The second is based on the use of matched filters.

The optimum detector that follows the signal demodulator is designed to minimize the probability of error.
Chapter 4.1-1
Optimal Detection for General Vector Channel
4.1.1 Optimal Detection for General Vector Channel (1)

- **AWGN channel model:** \( \mathbf{r} = \mathbf{s}_m + \mathbf{n} \)
  - Message \( m \) is chosen from the set \( \{1, 2, \ldots, M\} \) with probability \( P_m \)
  - Components in \( \mathbf{n} \) are i.i.d. \( N(0, N_0/2) \); PDF of noise \( \mathbf{n} \) is
    \[
    p(n) = \left( \frac{1}{\sqrt{\pi N_0}} \right)^N e^{-\frac{\sum_{j=1}^{N} n_j^2}{2\sigma^2}} = \left( \frac{1}{\sqrt{\pi N_0}} \right)^N e^{-\frac{\|n\|^2}{N_0}}
    \]

- A more general vector channel model:
  - \( s_m \) is selected from \( \{s_m, 1 \leq j \leq M\} \) according to *a priori* probability \( P_m \)
  - The received vector \( \mathbf{r} \) statistically depends on the transmitted vector through the conditional PDF \( p(\mathbf{r}|s_m) \).
Based on the observation $r$, receiver decides which message is transmitted, $\hat{m} \in \{1, 2, \ldots, M\}$

**Decision function:** $g(r)$: a function from $R^N$ into messages $\{1, 2, \ldots, M\}$

Given $r$ is received, the probability of correct detection:

$$P[\text{correct decision} | r] = P[\hat{m} \text{ sent} | r]$$

The probability of correct detection

$$P[\text{correct decision}] = \int \int P[\text{correct decision}|r] p(r) dr = \int P[\hat{m} \text{ sent}|r] p(r) dr$$

Max $P[\text{correct detection}] = \max \ P[\hat{m} | r]$ for each $r$

**Optimal decision rule:**

$$\hat{m} = g_{opt}(r) = \arg \max_{1 \leq m \leq M} P[m | r]$$

$$= \arg \max_{1 \leq m \leq M} P[s_m | r]$$
MAP and ML Receivers

◊ Optimum decision rule:

\[
\hat{m} = g_{\text{opt}}(r) = \arg \max_{1 \leq m \leq M} P[s_m | r]
\]

⇒ **Maximize a posteriori probability (MAP) rule**

◊ **MAP rule can be simplified**

\[
\hat{m} = g_{\text{opt}}(r) = \arg \max_{1 \leq m \leq M} \frac{p(r, s_m)}{p(r)} = \arg \max_{1 \leq m \leq M} \frac{p(r | s_m) p(s_m)}{p(r)} = \arg \max_{1 \leq m \leq M} \frac{P_m p(r | s_m)}{p(r)}
\]

\[
= \arg \max_{1 \leq m \leq M} P_m p(r | s_m)
\]

◊ When the messages are equiprobable a priori, \(P_1 = \ldots = P_M = 1/M\)

\[
\hat{m} = g_{\text{opt}}(r) = \arg \max_{1 \leq m \leq M} p(r | s_m)
\]

◊ \(p(r | s_m)\) is called likelihood of message \(m\)

◊ **Maximum-likelihood (ML) receiver**

\[
\rightarrow \quad \text{Channel} \quad p(r | s_m) \quad \rightarrow \quad \text{Note: } p(r | s_m) \text{ is channel conditional PDF.}
\]
The Decision Region

- For any detector, \( R^N \rightarrow \{1, 2, \ldots, M\} \)
  \[ \Rightarrow \text{Partition the output space } R^N \text{ into } M \text{ regions } (D_1, D_2, \ldots, D_M), \]
  if \( r \in D_m \), then \( \hat{m} = g(r) = m \)
  - \( D_m \) is decision region for message \( m \)

- For a MAP detector,
  \[ D_m = \{ r \in R^N : P[m | r] > P[m' | r], \text{ for all } 1 \leq m' \leq M \text{ and } m' \neq m \} \]

- If more than one messages achieve the maximum a posteriori probability, arbitrary assign \( r \) to one of the decision regions
The Error Probability (1)

- When $s_m$ is transmitted, an error occurs when $r$ is not in $D_m$
- Symbol error probability of a receiver with $\{D_m, 1 \leq m \leq M\}$

$$P_e = \sum_{m=1}^{M} P_m P[r \notin D_m \mid s_m \text{ sent}] = \sum_{m=1}^{M} P_m P_{e|m}$$

- $P_{e|m}$ is the error probability when message $m$ is transmitted

$$P_{e|m} = \int_{D_m^c} p(r \mid s_m) dr = \sum_{1 \leq m' \leq M \atop m' \neq m} \int_{D_m^c} p(r \mid s_m) dr$$

- Symbol error probability (or message error probability)

$$P_e = \sum_{m=1}^{M} P_m \sum_{1 \leq m' \leq M \atop m' \neq m} \int_{D_m^c} p(r \mid s_m) dr$$
Another type of error probability: **Bit error probability** $P_b$: error probability in transmission of a single bit

- Requires knowledge of how different bit sequences are mapped to signal points
- Finding bit error probability is not easy unless the constellation exhibits certain symmetric property

Relation between symbol error prob. and bit error prob.

$$\frac{P_e}{k} \leq P_b \leq P_e \leq kP_b$$
**Sufficient Statistics (An Example)**

- Assumption (1): the observation \( r \) can be written in terms of \( r_1 \) and \( r_2, r = (r_1, r_2) \)

- Assumption (2): 
  \[
  p(r_1, r_2 \mid s_m) = p(r_1 \mid s_m) p(r_2 \mid r_1)
  \]

- The MAP detection becomes
  \[
  \hat{m} = \arg \max_{1 \leq m \leq M} P_m p(r \mid s_m) = \arg \max_{1 \leq m \leq M} P_m p(r_1, r_2 \mid s_m)
  \]
  \[
  = \arg \max_{1 \leq m \leq M} P_m p(r_1 \mid s_m) p(r_2 \mid r_1) = \arg \max_{1 \leq m \leq M} P_m p(r_1 \mid s_m)
  \]

- Under these assumptions, the optimal detection
  - Based only on \( r_1 \Rightarrow r_1 \): *sufficient statistics* for detection of \( s_m \)
  - \( r_2 \) can be ignored \( \Rightarrow r_2 \): *irrelevant data* or irrelevant information

- Recognizing sufficient statistics helps to reduce complexity of the detection through ignoring irrelevant data.
Assume that the receiver applies an invertible operation $G(r)$ before detection.

The optimal detection is

$$\hat{m} = \arg \max_{1 \leq m \leq M} P_m p(r, \rho \mid s_m) = \arg \max_{1 \leq m \leq M} P_m p(r \mid s_m) p(\rho \mid r)$$

$$= \arg \max_{1 \leq m \leq M} P_m p(r \mid s_m)$$

- when $r$ is given, $\rho$ does not depend on $s_m$.
- The optimal detector based on the observation of $\rho$ makes the same decision as the optimal detector based on the observation of $r$.
- The invertible does not change the optimality of the receiver.
Ex.4.1-3 Assume that the received vector is of the form

\[ r = s_m + n \]

where \( n \) is colored noise. Let us further assume that there existing an invertible whitening operator denoted by \( W \), s.t. \( v = Wn \) is a white vector.

Consider

\[ \rho = Wr = Ws_m + v \]

- Equivalent to a channel with white noise for detection
- No degradation on the performance
- The linear operator \( W \) is called a *whitening filter*
Chapter 4.2
Waveform and Vector AWGN Channels
Waveform AWGN channel:

\[ r(t) = s_m(t) + n(t) \]

- \( s_m(t) \in \{s_1(t), s_2(t), \ldots, s_M(t)\} \) with prior probability \( P_m \)
- \( n(t) \): zero-mean white Gaussian with PSD \( N_0/2 \)

By Gram-Schmidt procedure, we derive an orthonormal basis
\( \{\phi_j(t), 1 \leq j \leq N\} \), and vector representation of signals \( \{s_m, 1 \leq m \leq M\} \)

Noise process \( n(t) \) is decomposed into two components:

- \( n_1(t) = \sum_{j=1}^{N} n_j \phi_j(t) \), where \( n_j = \langle n(t), \phi_j(t) \rangle \)
- \( n_2(t) = n(t) - n_1(t) \)
Waveform & Vector AWGN Channels

- \( s_m(t) = \sum_{j=1}^{N} s_{mj} \phi_j(t) \), where \( s_{mj} = \langle s_m(t), \phi_j(t) \rangle \)

- \( r(t) = \sum_{j=1}^{N} (s_{mj} + n_j) \phi_j(t) + n_2(t) \)

- Define \( r_j = s_{mj} + n_j \)
  where 
  \( r_j = \langle s_m(t), \phi_j(t) \rangle + \langle n(t), \phi_j(t) \rangle = \langle s_m(t) + n(t), \phi_j(t) \rangle = \langle r(t), \phi_j(t) \rangle \)

- So \( r(t) = \sum_{j=1}^{N} r_j \phi_j(t) + n_2(t) \), where \( r_j = \langle r(t), \phi_j(t) \rangle \)

- The noise components \( \{n_j\} \) are iid zero-mean Gaussian with variance \( N_0/2 \)
Prove that the noise components \( \{n_j\} \) are iid zero-mean Gaussian with variance \( N_0/2 \)

\[
n_j = \int_{-\infty}^{\infty} n(t) \phi_j(t) dt
\]

\[
E[n_j] = E\left[ \int_{-\infty}^{\infty} n(t) \phi_j(t) dt \right] = \int_{-\infty}^{\infty} E[n(t)] \phi_j(t) dt = 0 \quad \text{(Zero-Mean)}
\]

\[
\text{COV}[n_i, n_j] = E[n_i n_j] - E[n_i]E[n_j] = E\left[ \int_{-\infty}^{\infty} n(t) \phi_i(t) dt \int_{-\infty}^{\infty} n(s) \phi_j(t) ds \right]
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[n(t)n(s)] \phi_i(t) \phi_j(s) dt ds
\]

\[
= (N_0/2) \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \delta(t-s) \phi_i(t) dt \right] \phi_j(s) ds
\]

\[
= (N_0/2) \int_{-\infty}^{\infty} \phi_i(s) \phi_j(s) ds = \begin{cases} N_0/2 & i = j \\ 0 & i \neq j \end{cases}
\]

\( n(t) \) is white.
\[ \text{COV}[n_j n_2(t)] = \mathbf{E}[n_j n_2(t)] = \mathbf{E}[n_j n(t)] - \mathbf{E}[n_j n_1(t)] \]

\[ = \mathbf{E} \left[ n(t) \int_{-\infty}^{\infty} n(s) \phi_j(s) ds \right] - \mathbf{E} \left[ n_j \sum_{i=1}^{N} n_i \phi_i(t) \right] \]

\[ = \frac{N_0}{2} \int_{-\infty}^{\infty} \delta(t - s) \phi_j(s) ds - \frac{N_0}{2} \phi_j(t) \]

\[ = \frac{N_0}{2} \phi_j(t) - \frac{N_0}{2} \phi_j(t) = 0 \]

\[ n_2(t) \text{ is uncorrelated with } \{n_j\} \]

\[ \Rightarrow n_2(t) \text{ and } n_1(t) \text{ are independent} \]
waveform & vector awgn channels

- Since is \( n_2(t) \) independent of \( s_m(t) \) and \( n_1(t) \)
- \( r(t) = \sum_{j=1}^{N} (s_{mj} + n_j) \phi_j(t) + n_2(t) \)
  - Only the first component carries information
  - Second component is irrelevant data and can be ignored

- The AWGN waveform channel
  \( r(t) = s_m(t) + n(t), \quad 1 \leq m \leq M \)

is equivalent to \( N \)-dimensional vector channel

\[ r = s_m + n, \quad 1 \leq m \leq M \]
Chapter 4.2-1
Optimal Detection for the Vector AWGN Channel
4.2-1 Optimal Detection for the Vector AWGN Channel (1)

The MAP detector in AWGN channel is

\[ \hat{m} = \arg \max_{1 \leq m \leq M} \left[ P_m p(r | s_m) \right] \]

\[ = \arg \max_{1 \leq m \leq M} P_m \left[ p_n(r - s_m) \right] \]

\[ = \arg \max_{1 \leq m \leq M} \left[ P_m \left( \frac{1}{\sqrt{\pi N_0}} \right)^N e^{-\frac{\|r-s_m\|^2}{N_0}} \right] \]

\[ = \arg \max_{1 \leq m \leq M} \left[ P_m e^{-\frac{\|r-s_m\|^2}{N_0}} \right] \]

\[ = \arg \max_{1 \leq m \leq M} \left[ \ln P_m - \frac{\|r-s_m\|^2}{N_0} \right] \]
4.2-1 Optimal Detection for the Vector AWGN Channel (2)

\[
\begin{align*}
\hat{m} &= \arg \max_{1 \leq m \leq M} \left[ \ln p_m - \frac{\|r - s_m\|^2}{N_0} \right] \\
&= \arg \max_{1 \leq m \leq M} \left[ \frac{N_0}{2} \ln p_m - \frac{1}{2} \|r - s_m\|^2 \right] \\
&= \arg \max_{1 \leq m \leq M} \left[ \frac{N_0}{2} \ln p_m - \frac{1}{2} \left( \|r\|^2 + \|s_m\|^2 - 2r \cdot s_m \right) \right] \\
&= \arg \max_{1 \leq m \leq M} \left[ \frac{N_0}{2} \ln p_m - \frac{1}{2} E_m + r \cdot s_m \right] \\
&= \arg \max_{1 \leq m \leq M} \left[ \eta_m + r \cdot s_m \right] \\
\end{align*}
\]

(MAP)

Multiply $N_0/2$

\[\|s_m\|^2 = E_m\]

$\|r\|^2$ is dropped

\[\eta_m = \frac{N_0}{2} \ln p_m - \frac{1}{2} E_m\]  

Bias term
4.2.1 Optimal Detection for the Vector AWGN Channel (3)

- **MAP** decision rule for AWGN vector channel:
  \[ \hat{m} = \arg \max_{1 \leq m \leq M} \left[ \eta_m + \mathbf{r} \cdot \mathbf{s}_m \right] \]
  \[ \eta_m = \frac{N_0}{2} \ln P_m - \frac{1}{2} E_m \]

- If \( P_m = 1/M \), for all \( m \), the optimal decision becomes
  \[ \hat{m} = \arg \max_{1 \leq m \leq M} \left[ \frac{N_0}{2} \ln P_m - \frac{1}{2} \| \mathbf{r} - \mathbf{s}_m \|^2 \right] = \arg \max_{1 \leq m \leq M} \left[ -\| \mathbf{r} - \mathbf{s}_m \|^2 \right] \]
  \[ = \arg \min_{1 \leq m \leq M} \| \mathbf{r} - \mathbf{s}_m \| \]

- **Nearest neighbor** or **minimum distance** detector

- Signals are **equiprobable** in **AWGN** channel

\( \rightarrow \) MAP=ML=minimum distance
4.2-1 Optimal Detection for the Vector AWGN Channel (4)

- For **minimum-distance** detector, boundary of decisions $D_m$ and $D_m'$ are equidistant from $s_m$ and $s_m'$.
- Right figure:
  - 2-dim constellation (N=2)
  - 4 signal points (M=4)
- When signals are equiprobable and have equal energy
  - $\eta_m = \frac{N_0}{2} \ln P_m - \frac{1}{2} E_m$ indep of $m$
  - $\hat{m} = \arg \max_{1 \leq m \leq M} r \cdot s_m$
4.2-1 Optimal Detection for the Vector AWGN Channel (5)

◊ In general, the decision region is
\[ D_m = \left\{ \mathbf{r} \in \mathbb{R}^N : \mathbf{r} \cdot \mathbf{s}_m + \eta_m > \mathbf{r} \cdot \mathbf{s}_{m'}, + \eta_{m'}, \text{ for all } 1 \leq m' \leq M \text{ and } m' \neq m \right\} \] (4.2-20)

◊ Each region is described in terms of at most M-1 inequalities

◊ For each boundary:
\[ \mathbf{r} \cdot (\mathbf{s}_m - \mathbf{s}_{m'}) > \eta_{m'} - \eta_m \quad \Rightarrow \text{equation of a hyperplane} \]

\[ \therefore \mathbf{r} \cdot \mathbf{s}_m = \int_{-\infty}^{\infty} r(t)s_m(t)dt \]
\[ E_m = \|\mathbf{s}_m\|^2 = \int_{-\infty}^{\infty} s_m^2(t)dt \]

\[ \therefore \text{ in AWGN channel,} \]

\[ \text{MAP : } \hat{m} = \arg \max_{1 \leq m \leq M} \left[ \frac{N_0}{2} \ln P_m + \int_{-\infty}^{\infty} r(t)s_m(t)dt - \frac{1}{2} \int_{-\infty}^{\infty} s_m^2(t)dt \right] \]

\[ \text{ML: } \hat{m} = \arg \max_{1 \leq m \leq M} \left[ \int_{-\infty}^{\infty} r(t)s_m(t)dt - \frac{1}{2} \int_{-\infty}^{\infty} s_m^2(t)dt \right] \]
4.2-1 Optimal Detection for the Vector AWGN Channel (6)

- **Distance metric**: Euclidean distance between \( r \) and \( s_m \)
  \[
  D(r, s_m) = \|r - s_m\|^2 = \int_{-\infty}^{\infty} (r(t) - s_m(t))^2 \, dt
  \]

- **Modified distance metric**: distance when \( \|r\|^2 \) is removed
  \[
  D'(r, s_m) = -2r \cdot s_m + \|s_m\|^2
  \]

- **Correlation metric**: negative of modified distance metric
  \[
  C(r, s_m) = 2r \cdot s_m - \|s_m\|^2 = 2\int_{-\infty}^{\infty} r(t)s_m(t)dt - \int_{-\infty}^{\infty} s_m^2(t)dt
  \]

- With these definitions,
  - **MAP**:
    \[
    \hat{m} = \arg \max_{1\leq m \leq M} \left[ N_0 \ln P_m - D(r, s_m) \right]
    = \arg \max_{1\leq m \leq M} \left[ N_0 \ln P_m + C(r, s_m) \right]
    \]
  - **ML**:
    \[
    \hat{m} = \arg \max_{1\leq m \leq M} C(r, s_m)
    \]
Optimal Detection for Binary Antipodal Signaling (1)

- In binary antipodal signaling,
  - $s_1(t) = s(t)$, with $p_1 = p$
  - $s_2(t) = -s(t)$, with $p_2 = 1 - p$
- Vector representation ($N=1$) is
  - $s_1 = \sqrt{E_s} = \sqrt{E_b}$
  - $s_2 = -\sqrt{E_s} = -\sqrt{E_b}$
- $D_1 = \left\{ r : r \sqrt{E_b} + \frac{N_0}{2} \ln p - \frac{1}{2} E_b > -r \sqrt{E_b} + \frac{N_0}{2} \ln(1 - p) - \frac{1}{2} E_b \right\}$
  - \[
  = \left\{ r : r > \frac{N_0}{4 \sqrt{E_b}} \ln \frac{1 - p}{p} \right\}
  = \{ r : r > r_{th} \}
  \]
  - $r_{th} = \frac{N_0}{4 \sqrt{E_b}} \ln \frac{1 - p}{p}$
Optimal Detection for Binary Antipodal Signaling (2)

\( D_1 = \left\{ r : r > r_{th} \equiv \frac{N_0}{4\sqrt{E_b}} \ln \frac{1-p}{p} \right\} \)

- When \( p \to 0, r_{th} \to \infty \), entire real line becomes \( D_2 \)
- When \( p \to 1, r_{th} \to -\infty \), entire real line becomes \( D_1 \)
- When \( p=1/2, r_{th} = 0 \), minimum distance rule

Error probability of MAP receiver:

\[
P_e = \sum_{m=1}^{2} \sum_{m' \neq m} P(m) \sum_{1 \leq m' \leq 2} \int_{D_m}^{} P(r \mid s_{m'}) dr
\]

\[
= p \int_{D_2}^{} p \left( r \mid s = \sqrt{E_b} \right) dr + (1-p) \int_{D_1}^{} p \left( r \mid s = -\sqrt{E_b} \right) dr
\]

\[
= p \int_{-\infty}^{r_{th}} p \left( r \mid s = \sqrt{E_b} \right) dr + (1-p) \int_{r_{th}}^{\infty} p \left( r \mid s = -\sqrt{E_b} \right) dr
\]
Optimal Detection for Binary Antipodal Signaling (3)

\[ P_e = p \int_{\infty}^{r_{th}} p\left( r \mid s = \sqrt{E_b} \right) dr + (1 - p) \int_{r_{th}}^{\infty} p\left( r \mid s = -\sqrt{E_b} \right) dr \]

\[ = pP\left[ N\left( \sqrt{E_b}, N_0 / 2 \right) < r_{th} \right] + (1 - p)P\left[ N\left( -\sqrt{E_b}, N_0 / 2 \right) > r_{th} \right] \]

\[ = pQ\left( \frac{\sqrt{E_b} - r_{th}}{\sqrt{N_0 / 2}} \right) + (1 - p)Q\left( \frac{r_{th} + \sqrt{E_b}}{\sqrt{N_0 / 2}} \right) \]

\[ \text{When } p=1/2, \ r_{th} =0, \text{ the error probability is simplified as} \]

\[ P_e = Q\left( \sqrt{\frac{2E_b}{N_0}} \right) \]

\[ \text{Since the system is binary, } P_e = P_b \]

\[ P\left[ N\left( \sqrt{E_b}, N_0 / 2 \right) < r_{th} \right] = 1 - P\left[ N\left( \sqrt{E_b}, N_0 / 2 \right) > r_{th} \right] \]

\[ = 1 - Q\left( \frac{r_{th} - \sqrt{E_b}}{\sqrt{N_0 / 2}} \right) \]

\[ = Q\left( \frac{\sqrt{E_b} - r_{th}}{\sqrt{N_0 / 2}} \right) \]
In AWGN channel, transmitter transmits either \( s_1(t) \) or \( s_2(t) \), and assume two signals are *equiprobable*

- Equiprobable in AWGN channel \( \rightarrow \) decision regions are separated by the prependicular bisector of the line connecting \( s_1 \) and \( s_2 \)
- Error probabilities when \( s_1 \) or \( s_2 \) is transmitted are equal
- When \( s_1 \) is transmitted, error occurs when
  - \( r \) is in \( D_2 \)
  - Distance between the projection of \( r-s_1 \) on \( s_2-s_1 \) from \( s_1 \) is greater than \( d_{12}/2 \), \( d_{12} = \| s_2-s_1 \| \)

\[
P_b = P \left[ \frac{\mathbf{n} \cdot (s_2 - s_1)}{d_{12}} > \frac{d_{12}}{2} \right] = P \left[ \mathbf{n} \cdot (s_2 - s_1) > \frac{d_{12}^2}{2} \right]
\]

\[
\frac{s_2 - s_1}{d_{12}} \text{ is a unit vector. } \mathbf{n} = r - s_1.
\]
Error Probability for Equiprobable Binary Signaling Schemes (2)

- Since \( n \cdot (s_2 - s_1) \sim N(0, d_{12}^2 N_0 / 2) \)

\[
P_b = P\left[ n \cdot (s_2 - s_1) > d_{12}^2 / 2 \right] = Q\left( \frac{d_{12}^2 / 2}{d_{12} \sqrt{N_0 / 2}} \right) = Q\left( \sqrt{\frac{d_{12}^2}{2N_0}} \right)
\]

- Since \( Q(x) \) is decreasing, min. error probability = max. \( d_{12} \)

\[
d_{12}^2 = \int_{-\infty}^{\infty} (s_1(t) - s_2(t))^2 \, dt
\]

- When equiprobable signals have the same energy, \( E_{s_1} = E_{s_2} = E \)

\[
d_{12}^2 = E_{s_1} + E_{s_2} - 2 \langle s_1(t), s_2(t) \rangle = 2E(1 - \rho)
\]

- \(-1 \leq \rho \leq 1\) is cross-correlation coefficient

- \( d_{12} \) is maximized when \( \rho = -1 \) \(\Rightarrow\) antipodal signals

\[
P[X > \alpha] = Q\left( \frac{\alpha - m}{\sigma} \right) \quad (2.3-12)
\]

\[
P[X < \alpha] = Q\left( \frac{m - \alpha}{\sigma} \right)
\]
For binary orthogonal signals,
\[
\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \begin{cases} 
E_b & i = j \\
0 & i \neq j 
\end{cases} \quad 1 \leq i, j \leq 2
\]

Choose \( \phi_j(t) = \frac{s_j(t)}{\sqrt{E_b}} \), vector representation is
\[
s_1 = (\sqrt{E_b}, 0); \quad s_2 = (0, \sqrt{E_b})
\]

When signals are equiprobable (figure)

- Error probability : (\( d = \sqrt{2E_b} \))
  \[P_b = Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)\]

- Given the same \( P_b \), binary orthogonal signals requires twice energy of antipodal signals

- A binary orthogonal signaling requires twice the energy per bit of a binary antipodal signaling system to provide the same error probability.
Figure: Error Probability v.s. SNR/bit for binary orthogonal and binary antipodal signaling systems.

Signal-to-noise ratio (SNR) per bit

\[ \gamma_b = \frac{E_b}{N_0} \]
Chapter 4.2-2
Implementation of the Optimal Receiver for AWGN Channels

Wireless Information Transmission System Lab.
Institute of Communications Engineering
National Sun Yat-sen University
Present different implementations of MAP receivers in AWGN channel

- Correlation Receiver
- Matched Filter Receiver

All these structures are equivalent in performance and result in minimum error probability.
The MAP decision in AWGN channel (from 4.2-17)

\[ \hat{m} = \arg \max_{1 \leq m \leq M} [\eta_m + r \cdot s_m], \quad \text{where} \quad \eta_m = \frac{N_0}{2} \ln P_m - \frac{1}{2} E_m \]

1) \( r \) is derived from
\[ r_j = \int_{-\infty}^{\infty} r(t) \phi_j(t) dt \quad \rightarrow \text{Correlation receiver} \]

1) Find the inner product of \( r \) and \( s_m, \ 1 \leq m \leq M \)
2) Add the bias term \( \eta_m \)
3) Choose \( m \) that maximize the result
The Correlation Receiver (2)

$\eta_m$'s and $s_m$'s can be computed once and stored in memory.
Another implementation

\[ \hat{m} = \underset{1 \leq m \leq M}{\arg \max}[\eta_m + \int_{-\infty}^{\infty} r(t)s_m(t)dt], \quad \text{where } \eta_m = \frac{N_0}{2} \ln P_m - \frac{1}{2} E_m \]

- Requires \( M \) correlators
- Usually \( M > N \)
- Less preferred
The Matched Filter Receiver (1)

- In both correlation receiver, we compute
  \[ r_x = \int_{-\infty}^{\infty} r(t)x(t)dt \]

- \( x(t) \) is \( \phi_j(t) \) or \( s_m(t) \)

- Define \( h(t) = x(T-t) \) for arbitrary \( T \): filter matched to \( x(t) \)

- If \( r(t) \) is applied to \( h(t) \), the output \( y(t) \) is
  \[ y(t) = r(t)*h(t) = \int_{-\infty}^{\infty} r(\tau)h(t-\tau)d\tau \]
  \[ = \int_{-\infty}^{\infty} r(\tau)x(T-t+\tau)d\tau \]

- \( r_x = y(T) = \int_{-\infty}^{\infty} r(\tau)x(\tau)d\tau \)

- \( r_x \) can be obtained by sampling the output of matched filter at \( t=T \).
A matched filter implementation of the optimal receiver
Frequency Domain Interpretation:

- **Property I:**
  - Matched filter to signal $s(t)$ is $h(t)=s(T-t)$. The properties of Fourier transform is
    \[ H(f) = S^*(f)e^{-j2\pi fT} \]
    
    \[ |H(f)| = |S(f)| \]
    \[ \angle H(f) = -\angle S(f) - 2\pi fT \]
Property II: signal-to-noise maximizing property

Assume that \( r(t) = s(t) + n(t) \) is passed through a filter \( h(t) \), and the output \( y(t) = y_s(t) + \nu(t) \) is sampled at time \( T \)

Signal Part: \( F\{ y_s(t) \} = H(f) S(f) \)

\[ y_s(T) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT} df \]

Zero-Mean Gaussian Noise: \( S_\nu(f) = (N_0/2)|H(f)|^2 \)

\[ \text{VAR}[\nu(T)] = (N_0/2)\int_{-\infty}^{\infty}|H(f)|^2 df = (N_0/2)E_h \]

\( E_h \) is the energy in \( h(t) \).

Rayleigh's Theorem:

\[ \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = E_x \]
The SNR at the output of filter $H(f)$ is
\[
\text{SNR}_o = \frac{y_s^2(T)}{\text{VAR}[\nu(T)]}
\]

From Cauchy-Schwartz inequality
\[
y_s^2(T) = \left[ \int_{-\infty}^{\infty} H(f)S(f) e^{j2\pi fT} df \right]^2 \leq \left[ \int_{-\infty}^{\infty} |H(f)|^2 df \right] \left[ \int_{-\infty}^{\infty} |S(f) e^{j2\pi fT}|^2 df \right] = E_h E_s
\]

Equality holds iff $H(f) = \alpha S^*(f) e^{-j2\pi fT}$, $\alpha \in \mathbb{C}$

\[
\text{SNR}_o \leq \frac{E_s E_h}{(N_0/2)E_h} = \frac{2E_s}{N_0} = \frac{E_s}{N_0/2}
\]

The matched filter $h(t) = s(T-t)$, i.e. $H(f) = S^*(f) e^{-j2\pi fT}$, maximizes SNR
Matched-Filter

- Time-Domain Property of the matched filter.
  - If a signal $s(t)$ is corrupted by AWGN, the filter with an impulse response matched to $s(t)$ maximizes the output signal-to-noise ratio (SNR).
  - Proof: let us assume the receiver signal $r(t)$ consists of the signal $s(t)$ and AWGN $n(t)$ which has zero-mean and
    \[ \Phi_{nm}(f) = \frac{1}{2} N_0 \text{ W/Hz}. \]
  - Suppose the signal $r(t)$ is passed through a filter with impulse response $h(t)$, $0 \leq t \leq T$, and its output is sampled at time $t=T$. The output signal of the filter is:
    \[
y(t) = r(t) * h(t) = \int_{0}^{t} r(\tau)h(t - \tau) d\tau \\
    = \int_{0}^{t} s(\tau)h(t - \tau) d\tau + \int_{0}^{t} n(\tau)h(t - \tau) d\tau
    \]
Proof: (cont.)

At the sampling instant \( t = T \):

\[
y(T) = \int_0^T s(\tau)h(T - \tau)d\tau + \int_0^T n(\tau)h(T - \tau)d\tau
\]

\[
= y_s(T) + y_n(T)
\]

This problem is to select the filter impulse response that maximizes the output SNR \( \text{SNR}_0 \) defined as:

\[
\text{SNR}_0 = \frac{y_s^2(T)}{E\left[y_n^2(T)\right]}
\]

\[
E\left[y_n^2(T)\right] = \int_0^T \int_0^T E\left[n(\tau)n(t)\right] h(T - \tau)h(T - t)dtd\tau
\]

\[
= \frac{1}{2} N_0 \int_0^T \int_0^T \delta(t - \tau)h(T - \tau)h(T - t)dtd\tau = \frac{1}{2} N_0 \int_0^T h^2(T - t)dt
\]
Proof: (cont.)

- By substituting for $y_s(T)$ and $E\left[ y_n^2(T) \right]$ into $\text{SNR}_0$.

\[ \tau' = T - \tau \]

\[
\text{SNR}_0 = \frac{\left[ \int_0^T s(\tau)h(T-\tau)d\tau \right]^2}{\frac{1}{2} N_0 \int_0^T h^2(T-t)dt} = \frac{\int_0^T h(\tau')s(T-\tau')d\tau'}{\frac{1}{2} N_0 \int_0^T h^2(T-t)dt}
\]

- Denominator of the SNR depends on the energy in $h(t)$.
- The maximum output SNR over $h(t)$ is obtained by maximizing the numerator subject to the constraint that the denominator is held constant.
Proof: (cont.)

- **Cauchy-Schwarz inequality**: if \( g_1(t) \) and \( g_2(t) \) are finite-energy signals, then

\[
\left[ \int_{-\infty}^{\infty} g_1(t) g_2(t) dt \right]^2 \leq \int_{-\infty}^{\infty} g_1^2(t) dt \int_{-\infty}^{\infty} g_2^2(t) dt
\]

with equality when \( g_1(t) = C g_2(t) \) for any arbitrary constant \( C \).

- If we set \( g_1(t) = h_1(t) \) and \( g_2(t) = s(T-t) \), it is clear that the SNR is maximized when \( h(t) = Cs(T-t) \).
Proof: (cont.)

The output (maximum) SNR obtained with the matched filter is:

\[
\text{SNR}_0 = \frac{\left( \int_{0}^{T} s(\tau)h(T - \tau)d\tau \right)^2}{\frac{1}{2}N_0 \int_{0}^{T} h^2(T - t)dt} = \frac{2}{N_0} \left[ \int_{0}^{T} s(\tau)Cs(T - (T - \tau))d\tau \right]^2
\]

\[
= \frac{2}{N_0} \int_{0}^{T} s^2(t)dt = \frac{2\mathcal{E}}{N_0}
\]

Note that the output SNR from the matched filter depends on the energy of the waveform \(s(t)\) but not on the detailed characteristics of \(s(t)\).
Ex 4.2-1 M=4 biorthogonal signals are constructed by two signals in fig.(a) for transmission in AWGN channel. The noise is zero mean and PSD=$N_0/2$.

Dimension $N=2$, and basis function

\[
\phi_1(t) = \sqrt{2/T}, \quad 0 \leq t \leq T/2
\]
\[
\phi_2(t) = \sqrt{2/T}, \quad T/2 \leq t \leq T
\]

Impulse response of two matched filters (fig (b))

\[
h_1(t) = \phi_1(T-t) = \sqrt{2/T}, \quad T/2 \leq t \leq T
\]
\[
h_2(t) = \phi_2(T-t) = \sqrt{2/T}, \quad 0 \leq t \leq T/2
\]

\[
y(t) = s(t) * h(t) = \int_{-\infty}^{\infty} s(\tau) h(t-\tau) d\tau
\]
\[
= \int_{-\infty}^{\infty} s(\tau) s(T-t+\tau) d\tau
\]
The Matched Filter Receiver (7)

- If $s_1(t)$ is transmitted, the noise-free responses of two matched filter\((\text{fig\,(c)})\) sampled at $t=T$ are
  \[ y_{1s}(T) = \sqrt{A^2T/2} = \sqrt{E}; \quad y_{2s}(T) = 0 \]

- If $s_1(t)$ is transmitted, the received vector formed from two matched filter outputs at sampling instances $t = T$ is
  \[ r = (r_1, r_2) = (\sqrt{E} + n_1, n_2) \]

- Noise: $n_1 = y_{1n}(T)$ & $n_2 = y_{2n}(T)$
  \[ y_{kn}(T) = \int_0^T n(t)\phi_k(t)dt, \quad k = 1, 2 \]

a) $E[n_k] = E[y_{kn}(T)] = 0$

b) $\text{VAR}[n_k] = (N_0/2)E_{\phi_k} = N_0/2$

- SNR for the first matched filter
  \[ \text{SNR}_0 = \frac{(\sqrt{E})^2}{N_0/2} = \frac{2E}{N_0} \]
Chapter 4.2-3
A Union Bound on Probability of Errors of ML Detection

Wireless Information Transmission System Lab.
Institute of Communications Engineering
National Sun Yat-sen University
When signals are equiprobable, $P_m = 1/M$, ML decision is optimal. The error probability becomes

$$P_e = \frac{1}{M} \sum_{m=1}^{M} P_{e|m} = \frac{1}{M} \sum_{m=1}^{M} \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p(r | s_m) dr$$

For AWGN channel,

$$P_{e|m} = \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p(r | s_m) dr = \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p_n(r - s_m) dr = \left(\frac{1}{\sqrt{\pi N_0}}\right)^N \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} e^{-\frac{|r - s_m|^2}{N_0}} dr$$

For most constellation, the integrals does not have a close form

It’s convenient to have upper bounds for the error probability

The union bound is the simplest and most widely used bound which is quite tight particularly at high SNR.
A Union Bound on Probability of Errors of ML Detection (2)

- In general, the decision region \( D_{m'} \) under ML detection is
  \[
  D_{m'} = \{ r \in \mathbb{R}^N : p(r | s_{m'}) > p(r | s_k), \quad \text{for all } 1 \leq k \leq M \text{ and } k \neq m' \}
  \]

- Define \( D_{mm'} = \{ p(r | s_{m'}) > p(r | s_m) \} \)

- Decision region for \( m' \) in a binary equiprobable signals \( s_m \) & \( s_{m'} \)
  \[
  D_{m'} \subseteq D_{mm'}
  \]

- Pairwise Error Probability \( P_{m \rightarrow m'} \)
  \[
  \int_{D_{m'}} p(r | s_m) dr \leq \int_{D_{mm'}} p(r | s_m) dr \tag{4.2-67}
  \]

- Decision region for \( m' \) in a binary equiprobable signals \( s_m \) & \( s_{m'} \)

- \[
  P_{e|m} = \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p(r | s_{m'}) dr \leq \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{mm'}} p(r | s_{m'}) dr = \sum_{1 \leq m' \leq M, m' \neq m} P_{m \rightarrow m'}
  \]

- \[
  P_e \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{mm'}} p(r | s_m) dr = \frac{1}{M} \sum_{m=1}^{M} \sum_{1 \leq m' \leq M, m' \neq m} P_{m \rightarrow m'}
  \]
4.2-3
A Union Bound on Probability of Errors of ML Detection (3)

◊ In an AWGN channel

◊ Pairwise probability: \( P_{m \rightarrow m'} = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{\sqrt{2N_0}} \left( 1 - Q\left( \frac{d_{mm'}^2}{2N_0} \right) \right) \)  (4.2-37)

\[ P_e \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{m' \neq m} e^{-\frac{d_{mm'}^2}{4N_0}} \]

◊ Distance enumerator function for a constellation \( T(X) \):

\[ T(X) = \sum_{d_{mm'} = ||s_m - s_{m'}||, 1 \leq m, m' \leq M, m \neq m'} X^{d_{mm'}} = \sum a_d X^{d^2} \]

◊ \( a_d \): # of ordered pairs \((m,m')\) s.t. \( m \neq m' \), and \( ||s_m - s_{m'}|| = d \)
4.2-3
A Union Bound on Probability of Errors of ML Detection (4)

◊ **Union bound:**
\[
P_e \leq \frac{1}{2M} \sum_{m=1}^{M} \sum_{1 \leq m' \leq M, m' \neq m} e^{-\frac{d_{mm'}^2}{4N_0}} = \frac{1}{2M} T(X) \bigg|_{X=e^{-\frac{1}{4N_0}}}
\]

◊ **Minimum distance:**
\[
d_{\text{min}} = \min_{1 \leq m, m' \leq M, m \neq m'} \| s_m - s_{m'} \|
\]

◊ Since \( Q(x) \) is decreasing
\[
Q\left( \sqrt{\frac{d_{mm'}^2}{2N_0}} \right) \leq Q\left( \sqrt{\frac{d_{\text{min}}^2}{2N_0}} \right)
\]

◊ The error probability (looser form of the union bound)
\[
P_e \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{1 \leq m' \leq M, m' \neq m} Q\left( \sqrt{\frac{d_{mm'}^2}{2N_0}} \right) \leq (M - 1) Q\left( \sqrt{\frac{d_{\text{min}}^2}{2N_0}} \right) \leq \frac{M - 1}{2} e^{-\frac{d_{\text{min}}^2}{4N_0}}
\]

◊ Good constellation provides maximum possible minimum distance
Ex 4.2-2 Consider a 16-QAM constellation (M=16)

From Chap 3.2(equation 3.2-44), the minimum distance is

\[ d_{\text{min}} = \sqrt{\frac{6 \log_2 M}{M - 1}} E_{\text{bavg}} = \sqrt{\frac{8}{5}} E_{\text{bavg}} \]

- Total 16x15=240 possible distances
- Distance enumerator function:

\[ T(X) = 48X^{d^2} + 36X^{2d^2} + 32X^{4d^2} + 48X^{5d^2} + 16X^{8d^2} + 16X^{9d^2} + 24X^{10d^2} + 16X^{13d^2} + 4X^{18d^2} \]

- Upper bound of error probability

\[ P_e \leq \frac{1}{32} T \left( e^{-\frac{1}{4N_0}} \right) \]
A Union Bound on Probability of Errors of ML Detection (6)

\[ P_e \leq \frac{M - 1}{2} e^{-\frac{d_{\text{min}}^2}{4N_0}} = \frac{15}{2} e^{-\frac{2E_{\text{bavg}}}{5N_0}} \]

\[ \text{When SNR is large } \left( T(X) \approx 48X^d \right), \]

\[ P_e \leq \frac{1}{32} T \left( e^{-\frac{1}{4N_0}} \right) \approx \frac{48}{32} e^{-\frac{d_{\text{min}}^2}{4N_0}} = \frac{3}{2} e^{-\frac{2E_{\text{bavg}}}{5N_0}} \]

\[ \text{Exact error probability} \]

\[ P_e = 3Q \left( \frac{\sqrt{4E_{\text{bavg}}}}{5N_0} \right) - \frac{9}{4} \left[ Q \left( \frac{\sqrt{4E_{\text{bavg}}}}{5N_0} \right) \right]^2 \]

(see example 4.3-1)
In an equiprobable $M$-ary signaling scheme

$$P_e = \frac{1}{M} \sum_{m=1}^{M} P[\text{Error} \mid m \text{ sent}] = \frac{1}{M} \sum_{m=1}^{M} \int_{D_m} p(r \mid s_m) dr$$

$$\geq \frac{1}{M} \sum_{m=1}^{M} \int_{D_m'} p(r \mid s_m) dr = \frac{1}{M} \sum_{m=1}^{M} \int_{D_{mm'}} p(r \mid s_m) dr$$

$$= \frac{1}{M} \sum_{m=1}^{M} Q\left(\frac{d_{mm'}}{\sqrt{2N_0}}\right), \quad \text{for all } m' \neq m$$

(Finding $m'$ such that $d_{mm'}$ is minimized.)

To derive the tightest lower bound maximize the right hand side

$$P_e \geq \frac{1}{M} \sum_{m=1}^{M} \max_{m' \neq m} Q\left(\frac{d_{mm'}}{\sqrt{2N_0}}\right) = \frac{1}{M} \sum_{m=1}^{M} Q\left(\frac{d_{\text{min}}^m}{\sqrt{2N_0}}\right)$$

$d_{\text{min}}^m$ : distance from $m$ to its nearest neighbor
Lower Bound on Probability of Error (2)

\[ \diamond \therefore d_{\min}^m \geq d_{\min} \]
\[ \text{(}d_{\min}^m \text{ denotes the distance from } m \text{ to its nearest neighbor in the constellation)} \]
\[ Q \left( \frac{d_{\min}^m}{\sqrt{2N_0}} \right) \geq \begin{cases} Q \left( \frac{d_{\min}}{\sqrt{2N_0}} \right), & \text{At least one signal at distance } d_{\min} \text{ from } s_m \\ 0, & \text{otherwise} \end{cases} \]

\[ \diamond P_e \geq \frac{1}{M} \sum_{1<m<M} Q \left( \frac{d_{\min}}{\sqrt{2N_0}} \right) \]

\[ \exists m' \neq m: \|s_m - s_{m'}\| = d_{\min} \]

\[ \diamond N_{\min} : \text{number of points in the constellation s.t.} \]
\[ \exists m' \neq m : \|s_m - s_{m'}\| = d_{\min} \]

\[ \frac{N_{\min}}{M} Q \left( \frac{d_{\min}}{\sqrt{2N_0}} \right) \leq P_e \leq (M - 1)Q \left( \frac{d_{\min}}{\sqrt{2N_0}} \right) \]
Chapter 4.3: Optimal Detection and Error Probability for Bandlimited Signaling
In this section we study signaling schemes that are mainly characterized by their low bandwidth requirements.

These signaling schemes have low dimensionality which is independent from the number of transmitted signals, and, as we will see, their power efficiency decreases when the number of messages increases.

This family of signaling schemes includes ASK, PSK, and QAM.
For ASK signaling scheme:

\[
d_{\text{min}} = \sqrt{\frac{12 \log_2 M}{M^2 - 1} E_{\text{avg}}}
\] (3.2-22)

The constellation points: \( \{ \pm d_{\text{min}}/2, \pm 3d_{\text{min}}/2, \ldots, \pm (M-1)d_{\text{min}}/2 \} \)

Two type of signal points

- \( M-2 \) inner points: detection error occurs if \(|n| > d_{\text{min}}/2\).
- 2 outer points: error probability is half of inner points.

Let \( P_{ei} \) and \( P_{eo} \) are error probabilities of inner and outer points

\[
P_{ei} = P\left[ |n| > \frac{1}{2} d_{\text{min}} \right] = 2Q\left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right)
\]
\[
P_{eo} = \frac{1}{2} P_{ei} = Q\left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right)
\]
Symbol error probability

\[ P_e = \frac{1}{M} \sum_{m=1}^{M} P[\text{error} \mid m \text{ sent}] = \frac{1}{M} \left[ 2(M - 2)Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right) + 2Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right)\right] \]

\[ = \frac{2(M - 1)}{M} Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right) \]

\[ = 2\left(1 - \frac{1}{M}\right) Q\left(\sqrt{\frac{6\log_2 M E_{\text{avg}}}{M^2 - 1}}\right) \]

\[ \approx 2Q\left(\sqrt{\frac{6\log_2 M E_{\text{avg}}}{M^2 - 1}}\right) \text{ for large } M \]

\[ d_{\text{min}} = \sqrt{\frac{12 \log_2 M E_{\text{avg}}}{M^2 - 1}} \]

Doubling \( M \)

- increasing rate by 1 bit/transmission
- Need 4 times SNR/bit to keep performance

Decrease with \( M \)

SNR/bit
Symbol error probability for ASK or PAM signaling

For large $M$, the distance between $M$ and $2M$ is roughly 6dB.
In $M$-ary PSK signaling, assume signals are equiprobable.

- minimum distance decision is optimal.
- error probability = error prob. when $s_1$ is transmitted.

When $s_1 = (\sqrt{E}, 0)$ is transmitted, received signal is $r = (r_1, r_2) = (\sqrt{E} + n_1, n_2)$

- $r_1 \sim N(\sqrt{E}, N_0/2)$ and $r_2 \sim N(0, N_0/2)$ are indep.

$$p(r_1, r_2) = \frac{1}{\pi N_0} \exp \left( -\frac{(r_1 - \sqrt{E})^2 + r_2^2}{N_0} \right)$$

$$V = \sqrt{r_1^2 + r_2^2}, \ \Theta = \arctan \frac{r_2}{r_1}$$

$$p_{v, \Theta}(v, \Theta) = \frac{v}{\pi N_0} \exp \left( -\frac{v^2 + E - 2\sqrt{E}v \cos \Theta}{N_0} \right)$$
◊ Marginal PDF of Q is

\[ p_\theta(\theta) = \int_0^\infty p_{V,\theta}(v, \theta) dv \]

\[ = \int_0^{\infty} \frac{\nu}{\pi N_0} e^{-\frac{\nu^2 + E - 2\sqrt{E}v\cos\theta}{N_0}} dv \]

\[ = \frac{1}{2\pi} e^{-\gamma_s \sin^2 \theta} \int_0^{\infty} ve^{-\frac{(v-\sqrt{2\gamma_s \cos\theta})^2}{2}} dv \]

◊ Symbol SNR: \( \gamma_s = \frac{E}{N_0} \)

◊ As \( g_s \) increases, \( p_Q(q) \) is more peaked around \( q = 0 \).
Chapter 4.3-2  Optimal Detection & Error Prob. for PSK Signaling

◊ Decision region: \( D_1 = \{\theta : -\pi / M < \theta \leq \pi / M\}\)

◊ Error probability is

\[
P_e = 1 - \int_{-\pi / M}^{\pi / M} p_{\Theta}(\theta) d\theta
\]

◊ Not have a simple form except for \( M=2 \) or \( 4 \).

◊ When \( M=2 \) ➞ binary antipodal signaling

\[
P_b = Q\left(\sqrt{2E_b / N_0}\right)
\]

◊ When \( M=4 \), ➞ two binary phase modulation signals

\[
P_c = (1 - P_b)^2 = \left[1 - Q\left(\sqrt{2E_b / N_0}\right)\right]^2
\]

\[
P_e = 1 - P_c = 2Q\left(\sqrt{2E_b / N_0}\right)\left[1 - \frac{1}{2} Q\left(\sqrt{2E_b / N_0}\right)\right]
\]
Symbol error probability of M-PSK

\( M \uparrow, \text{Required SNR } \uparrow \)
For large SNR \((E/N_0 >> 1)\), \(p_\Theta(\theta)\) is approximated
\[
p_\Theta(\theta) \approx \sqrt{\gamma_s / \pi} \cos \theta \, e^{-\gamma_s \sin^2 \theta}, \text{ for } |\theta| \leq \pi / 2
\]

Error probability is approximated by
\[
P_e \approx 1 - \int_{-\pi/M}^{\pi/M} \sqrt{\gamma_s / \pi} \cos \theta \, e^{-\gamma_s \sin^2 \theta} \, d\theta
\]
\[
\approx \frac{2}{\sqrt{\pi}} \int_{-\gamma_s \sin(\pi/M)}^{\gamma_s \sin(\pi/M)} e^{-u^2} \, du
\]
\[
= 2Q\left(\sqrt{2\gamma_s \sin(\pi/M)}\right) = 2Q\left(\sqrt{(2 \log_2 M) \sin^2\left(\frac{\pi}{M}\right)} \frac{E_b}{N_0}\right)
\]

\[
\frac{E_b}{N_0} = \frac{E}{N_0 \log_2 M} = \frac{\gamma_s}{\log_2 M}
\]

When \(M=2\) or 4:
\[
P_e \approx 2Q\left(\sqrt{2E_b / N_0}\right)
\]
For large $M$ and large SNR:

- $\sin(\pi/M) \sim \pi/M$
- Error probability is approximated by

$$P_e \approx 2Q\left(\sqrt{\frac{2\pi^2 \log_2 M}{M^2} \frac{E_b}{N_0}}\right)$$

for large $M$

- For large $M$, doubling $M$ reduces effective SNR by 6 dB.

When Gray codes is used in the mapping

- Since the most probable errors occur in the erroneous selection of an adjacent phase to the true phase

$$P_b \approx \frac{1}{k} P_e$$
In practice, the carrier phase is extracted from the received signal by performing nonlinear operation \( \Rightarrow \) phase ambiguity

For BPSK
- The received signal is first squared.
- The resulting double-frequency component is filtered.
- The signal is divided by 2 in frequency to extract an estimate of the carrier frequency and phase \( \phi \).
- This operation result in a phase ambiguity of 180° in the carrier phase.

For QPSK, there are phase ambiguities of \( \pm 90° \) and 180° in the phase estimate.

Consequently, we do not have an absolute estimate of the carrier phase for demodulation.
Differentially Encoded PSK Signaling

- The phase ambiguity can be overcome by encoding the information in *phase differences* between successive signals.
  - In BPSK
    - Bit 1 is transmitted by shifting the phase of carrier by 180°.
    - Bit 0 is transmitted by a zero phase shift.
  - In QPSK, the phase shifts are 0, 90°, 180°, and -90°, corresponding to bits 00, 01, 11, and 10, respectively.

- The PSK signals resulting from the encoding process are *differentially encoded*.

- The detector is a simple *phase comparator* that compares the phase of the demodulated signal over two consecutive intervals to extract the information.
Differentially Encoded PSK Signaling

- Coherent demodulation of differentially encoded PSK results in a higher error probability than that derived for absolute phase encoding.
- With differentially encoded PSK, an error in the demodulated phase of the signal in any given interval usually results in decoding errors over two consecutive signaling intervals.
- The error probability in differentially encoded M-ary PSK is approximately twice the error probability of M-ary PSK with absolute phase encoding.
  - Only a relative small loss in SNR.
In detection of QAM signals, need two filter matched to
\[
\phi_1(t) = \sqrt{2 / E_g} g(t) \cos 2\pi f_c t
\]
\[
\phi_2(t) = -\sqrt{2 / E_g} g(t) \sin 2\pi f_c t
\]
Output of matched filters \( r = (r_1, r_2) \)
Compute \( C(r, s_m) = 2r \cdot s_m - E_m \) (See 4.2-28)
Select \( \hat{m} = \arg \max_{1 \leq m \leq M} C(r, s_m) \)

To determine \( P_e \) \( \Rightarrow \) must specify signal constellation
- For \( M=4 \), fig (a) and (b) are possible constellations.
- Assume both have \( d_{\min} = 2A \)
  - \( r = \sqrt{2A} \) \( \Rightarrow \) \( E_{avg} = 2A^2 \)
  - \( A_1 = A, A_2 = 3A \) \( \Rightarrow \) \( E_{avg} = \frac{1}{4}[2(3A^2) + 2A^2] = 2A^2 \)
When $M=8$, four possible constellations in fig (a)~(d).

- Signal points $(A_{mc}, A_{ms})$
- All have $d_{min} = 2A$
- Average energy

\[
E_{avg} = \frac{1}{M} \sum_{m=1}^{M} (A_{mc}^2 + A_{ms}^2) = \frac{A^2}{M} \sum_{m=1}^{M} (a_{mc}^2 + a_{ms}^2)
\]

- (a) and (C): $E_{avg} = 6A^2$
- (b): $E_{avg} = 6.83A^2$
- (d): $E_{avg} = 4.73A^2$
- (d) require less energy
Chapter 4.3-3 Optimal Detection & Error Prob. for QAM Signaling

◊ Rectangular QAM
  ◦ Generated by two PAM signals in I-phase and Q-phase carriers.
  ◦ Easily demodulated.
  ◦ For \( M \geq 16 \), only requires energy slight greater than that of the best 16-QAM constellation.
  ◦ When \( k \) is even, the constellation is square, the minimum distance
  \[
  d_{\text{min}} = \sqrt{\frac{6 \log_2 M}{M-1}} E_{\text{avg}}
  \]
  ◦ Can be considered as two \( \sqrt{M} \)-ary PAM constellations.
  ◦ An error occurs if either \( n_1 \) or \( n_2 \) is large enough to cause an error.
  ◦ Probability of correct decision is
  \[
P_{c,M-QAM} = P_{c,\sqrt{M}-PAM}^2 = \left(1 - P_{e,\sqrt{M}-PAM}\right)^2
  \]
Probability of errors of square M-QAM

\[ P_{e,M-QAM} = 1 - (1 - P_{e,\sqrt{M}-PAM})^2 = 2P_{e,\sqrt{M}-PAM}\left(1 - \frac{1}{2}P_{e,\sqrt{M}-PAM}\right) \]

The error probability of PAM is (from (4.3-4) & (4.3-5))

\[ P_{e,\sqrt{M}-PAM} = 2\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 2\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\log_2 M \cdot E_{bav}}{M - 1 \cdot N_0}}\right) \]

Thus, the error probability of square M-QAM is

\[ P_{e,M-QAM} = 4\left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\log_2 M \cdot E_{bav}}{M - 1 \cdot N_0}}\right) \times \left(1 - \frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\log_2 M \cdot E_{bav}}{M - 1 \cdot N_0}}\right) \leq \frac{1}{4} \]

\[ \leq 4Q\left(\sqrt{\frac{3\log_2 M \cdot E_{bav}}{M - 1 \cdot N_0}}\right) \]

The upper bound is quite tight for large M.
The penalty of increasing transmission rate is 3dB/bit for QAM.

The penalty of increasing transmission rate is 6dB/bit for PAM and PSK.

QAM is more power efficient compared with PAM and PSK.

The advantage of PSK is its constant-envelope properties.

More comparisons are shown in the text (page 200).
ASK, PSK and QAM have one- or two-dimensional constellation

- Basis functions of PSK and QAM:
  \[ \phi_1(t) = \sqrt{2 / E_g} g(t) \cos 2\pi f_c t \]
  \[ \phi_2(t) = -\sqrt{2 / E_g} g(t) \sin 2\pi f_c t \]

- Basis functions of PAM:
  \[ \phi_1(t) = \sqrt{2 / E_g} g(t) \cos 2\pi f_c t \]

- r(t) and basis functions are bandpass ➔ high sampling rate

- To relieve the requirement on sampling rate
  ➔ First, demodulate signals to obtain a lowpass equivalent signals.
  ➔ Then, perform signal detection.
From chap 2.1 (2.1-21 and 2.1-24):

\[ E_x = \frac{E_{x_i}}{2} \quad \langle x(t), y(t) \rangle = \text{Re}\{\langle x_i(t), y_i(t) \rangle}\] / 2

Optimal detection rule (MAP) becomes

\[
\hat{m} = \arg \max_{1 \leq m \leq M} \left( r \cdot s_m + \frac{N_0}{2} \ln P_m - \frac{1}{2} E_m \right)
\]

\[
= \arg \max_{1 \leq m \leq M} \left( \text{Re}[r \cdot s_{ml}] + N_0 \ln P_m - E_{ml} / 2 \right)
\]

\[
= \arg \max_{1 \leq m \leq M} \left( \text{Re} \left[ \int_{-\infty}^{\infty} r(t) s_{ml}^*(t) dt \right] + N_0 \ln P_m - \frac{1}{2} \int_{-\infty}^{\infty} |s_{ml}(t)|^2 dt \right)
\]

ML decision rule is

\[
\hat{m} = \arg \max_{1 \leq m \leq M} \left( \text{Re} \left[ \int_{-\infty}^{\infty} r(t) s_{ml}^*(t) dt \right] - \frac{1}{2} \int_{-\infty}^{\infty} |s_{ml}(t)|^2 dt \right)
\]
Complex matched filter.

Detailed structure of a complex matched filter in terms of its in-phase and quadrature components.

◊ Throughout this discussion we have assumed that the receiver has complete knowledge of the carrier frequency and phase.
Chapter 4.4: Optimal Detection and Error Probability for Power-Limited Signaling
In an *equal-energy* *orthogonal* signaling scheme, \( N=M \) and

\[
\begin{align*}
\mathbf{s}_1 &= (\sqrt{E}, 0, \ldots, 0) \\
\mathbf{s}_2 &= (0, \sqrt{E}, \ldots, 0) \\
\vdots &= \vdots \\
\mathbf{s}_M &= (0, 0, \ldots, \sqrt{E})
\end{align*}
\]

For *equiprobable* *equal-energy* *orthogonal* signals, optimum detector = largest cross-correlation between \( \mathbf{r} \) and \( \mathbf{s}_m \)

\[
\hat{m} = \arg \max_{1 \leq m \leq M} \mathbf{r} \cdot \mathbf{s}_m
\]

\[
\therefore \text{Constellation is symmetric & distance between signal points is } \sqrt{2E}
\]

\[
\therefore \text{Error probability is independent of the transmitted signal}
\]
Suppose that \( s_1 \) is transmitted, the received vector is

\[
r = (\sqrt{E} + n_1, n_2, \ldots, n_M)
\]

\( E \) is symbol energy

\( (n_1, n_2, \ldots, n_M) \) are iid zero-mean Gaussian r.v.s with \( \sigma_n^2 = N_0 / 2 \)

Define random variables

\[
R_m = r \cdot s_m, \quad 1 \leq m \leq M
\]

A correct decision is made if \( R_1 > R_m \) for \( m=2,3,\ldots, M \)

\[
P_c = P[R_1 > R_2, R_1 > R_3, \ldots, R_1 > R_M | s_1 \text{ sent}]
\]

\[
= P[\sqrt{E} + n_1 > n_2, \sqrt{E} + n_1 > n_3, \ldots, \sqrt{E} + n_1 > n_M | s_1 \text{ sent}]
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P[n_2 < n + \sqrt{E}, n_3 < n + \sqrt{E}, \ldots, n_M < n + \sqrt{E} | s_1 \text{ sent}, n_1 = n] \cdot p_{n_1}(n) \, dn
\]

\[
= \int_{-\infty}^{\infty} \left( P[n_2 < n + \sqrt{E} | s_1 \text{ sent}, n_1 = n] \right)^{M-1} p_{n_1}(n) \, dn
\]
Since $n_2 \sim N(0, N_0/2)$

$$P[n_2 < n + \sqrt{E} \mid s_1 \text{ sent}, n_1 = n] = 1 - Q\left(\frac{n + \sqrt{E}}{\sqrt{N_0/2}}\right)$$

$$P_e = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} \left[1 - Q\left(\frac{n + \sqrt{E}}{\sqrt{N_0/2}}\right)\right]^{M-1} e^{-\frac{n^2}{N_0}} dn$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 - Q(x))^{M-1} e^{-\frac{(x-\sqrt{2E/N_0})^2}{2}} dx$$

$$P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[1 - (1 - Q(x))^{M-1}\right] e^{-\frac{(x-\sqrt{2E/N_0})^2}{2}} dx$$

$$\int_{-\infty}^{\infty} e^{-\frac{(x-\sqrt{2E/N_0})^2}{2}} dx = 1$$

$$P[s_m \text{ received} \mid s_1 \text{ sent}] = \frac{P_e}{M-1} = \frac{P_e}{2^k-1}, \quad 2 \leq m \leq M.$$
Assume that $s_1$ corresponds to data sequence of length $k$ and first bit = 0

- Probability that first bit is detected as 1 = prob. of detecting as $\{s_m: \text{first bit}=1\}$
  \[
P_b = 2^{k-1} \frac{P_e}{2^k - 1} = 2^{k-1} \frac{P_e}{2^k - 1} \approx \frac{1}{2} P_e
  \]

- Last approximation holds for $k \gg 1$

- Fig: Prob. of bit error v.s. SNR per bit
  - Increasing $M$, required SNR is reduced
    - in contrast with ASK, PSK and QAM
钻的 FSK signaling is a special case of orthogonal signaling when

\[ \Delta f = \frac{l}{2T}, \quad \text{for any positive integer } l \]

diamond In binary FSK, a frequency separation that guarantees orthogonality does not minimize the error probability.

diamond For binary FSK, the error probability is minimized when (see Problem 4.18)

\[ \Delta f = \frac{0.715}{T} \]
From Sec.4.2-3, the union bound in AWGN channel is

\[ P_e \leq \frac{M - 1}{2} e^{\frac{d_{\text{min}}^2}{4N_0}} \]

In orthogonal signaling, \( d_{\text{min}} = \sqrt{2E} \)

\[ P_e \leq \frac{M - 1}{2} e^{\frac{E}{2N_0}} < Me^{\frac{E}{2N_0}} \]

Using \( M=2^k \) and \( E_b = E/k \),

\[ P_e < 2^k e^{\frac{kE_b}{2N_0}} = e^{-\frac{k}{2} \left( \frac{E_b}{N_0} - 2 \ln 2 \right)} \]

If \( E_b / N_0 > 2 \ln 2 = 1.39 \) (1.42dB) \[ \Rightarrow P_e \to 0 \text{ as } k \to \infty \]

⇒ If SNR per bit > 1.42 dB, reliable communication is possible (Sufficient, but not necessary)
◊ **A necessary and sufficient condition** for reliable communications is 
\[
\frac{E_b}{N_0} > \ln 2 = 0.693 \text{ (} -1.6 \text{dB)}
\]

◊ The -1.6 dB bound is obtained from a tighter bound on error probability
\[
P_e \leq \begin{cases} 
  e^{-(k/2)(E_b/N_0-2\ln 2)}, & E_b/N_0 > 4\ln 2 \\
  2e^{-k(\sqrt{E_b/N_0}-\sqrt{\ln 2})^2}, & \ln 2 \leq E_b/N_0 \leq 4\ln 2
\end{cases}
\]

◊ The minimum value of SNR per bit needed, i.e., -1.6 dB is **Shannon Limit**.
A set of $M=2^k$ biorthogonal signals comes from $N=M/2$ orthogonal signals by including the negatives of these signals. Requires only $M/2$ cross-correlators or matched filters.

Vector representation of biorthogonal signals

\[ s_1 = -s_{N+1} = (\sqrt{E}, 0, \ldots, 0) \]
\[ s_2 = -s_{N+2} = (0, \sqrt{E}, \ldots, 0) \]
\[ \vdots \]
\[ s_N = -s_{2N} = (0, 0, \ldots, \sqrt{E}) \]

Assume that $s_1$ is transmitted, the received signal vector is

\[ r = (\sqrt{E} + n_1, n_2, \ldots, n_N) \]

\{n_m\} are zero-mean, iid Gaussian r.vs with $\sigma_n^2 = N_0 / 2$
Since all signals are equiprobable and have equal energy, the optimum detector decides \( m \) with the largest magnitude of
\[
C(r, s_m) = r \cdot s_m, \quad 1 \leq m \leq M / 2
\]
The sign is to decide whether \( s_m(t) \) or \(-s_m(t)\) is transmitted

\[
P[\text{correct decision}] = P[r_1 = \sqrt{E} + n_1 > 0, r_1 > |r_m| = |n_m|, m = 2, 3, ..., M]
\]
But,

\[
P[|n_m| < r_1 | r_1 > 0] = \frac{1}{\sqrt{\pi N_0}} \int_{-r_1}^{r_1} e^{-x^2/N_0} dx = \frac{1}{\sqrt{2\pi}} \int_{N_0/2}^{N_0/2} e^{-x^2/2} dx
\]

Probability of correct decision is

\[
P_c = \int_0^\infty \left( \frac{1}{\sqrt{2\pi}} \int_{\sqrt{N_0/2}}^{r_1} e^{-x^2/2} dx \right) \left( \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2E/N_0}} e^{-x^2/2} dx \right)^{(M/2)-1} p(r_1) dr_1
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2E/N_0}}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-(v+\sqrt{2E/N_0})}^{v+\sqrt{2E/N_0}} e^{-x^2/2} dx \right) e^{-v^2/2} dv
\]

\( r_1 \sim N(\sqrt{E}, N_0/2) \)

\( v = r_1 \sqrt{2/N_0} \)
Symbol error Probability

\[ P_e = 1 - P_c \]

Fig: \( P_e \) v.s. \( E_b/N_0 \)

\[ E = k E_b \]
Simplex signals are obtained from shifting a set of orthogonal signals by the average of these orthogonal signals.

Geometry of simplex signals is *exactly the same* as that of original orthogonal signals.

The error probability equals to the original orthogonal signals.

Since simplex signals have lower energy, the energy in the expression of error probability should be scaled, i.e.,

$$P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ 1 - (1 - Q(x))^{M-1} \right] e^{-\left( x - \sqrt{\frac{M}{M-1} N_0} \right)^2 / 2} dx$$

A relative gain of $10 \log(M/M-1)$ dB over orthogonal signaling.

- $M=2 \rightarrow$ 3 dB gain; $M=10$, 0.46 dB gain
Chap 4.5: Optimal Detection in Presence of Uncertainty: Non-coherent Detection
Previous sections assume that signal \( \{s_m(t)\} \) or orthonormal basis \( \{\phi_j(t)\} \) are available.

In many cases the assumption is not valid:

- Transmission over channel introduces a random attenuation or a random phase shift to the signal.
- Imperfect knowledge of signals at rx when the tx and rx are not perfectly synchronized.
  - Although the tx knows \( \{s_m(t)\} \), due to asynchronism, it can only use \( \{s_m(t-t_d)\} \); \( t_d \) is random time slip between the tx and rx clock.

Consider transmission over AWGN channel with random parameter(\( \theta \))

\[
r(t) = s_m(t; \theta) + n(t)
\]

By K-L expansion theorem (2.8-2), we can find an orthonormal basis

\[
r = s_{m, \theta} + n
\]
The optimal (MAP) detection rule is (see 4.2-15)

\[ \hat{m} = \arg \max_{1 \leq m \leq M} P_m p(r \mid m) \]

\[ = \arg \max_{1 \leq m \leq M} P_m \int p(r \mid m, \theta) p(\theta) d\theta \]

\[ = \arg \max_{1 \leq m \leq M} P_m \int p_n (r - s_{m,\theta}) p(\theta) d\theta \]

The decision rule determines the decision regions

The minimum error probability is

\[ P_e = \sum_{m=1}^{M} P_m \int_{D_m^c} \left( \int p(r \mid m, \theta) p(\theta) d\theta \right) dr \]

\[ = \sum_{m=1}^{M} P_m \sum_{m' \neq m}^{M} \int_{D_{m'}} \left( \int p_n (r - s_{m,\theta}) p(\theta) d\theta \right) dr \quad (4.5-3) \]
(ex) Consider a binary antipodal signaling system w. equiprobable
signals \( s_1(t) = s(t) \) & \( s_2(t) = -s(t) \) in an AWGN channel w. noise PSD \( N_0/2 \)

- The channel is modeled as
  \[ r(t) = A s_m(t) + n(t) \]
- \( A > 0 \): random gain with PDF \( p(A) \)
- \( A < 0 \): random gain with PDF \( p(A) = 0 \)
- \( p(r \mid m, A) = p_n(r - A s_m) \)
- Optimal decision region for \( s_1(t) \) is

\[
D_1 = \left\{ r : \int_0^\infty e^{-(r-A\sqrt{E_b})^2/N_0} p(A) dA > \int_0^\infty e^{-(r+A\sqrt{E_b})^2/N_0} p(A) dA \right\}
\]

\[
= \left\{ r : \int_0^\infty e^{-A^2 E_b / N_0} \left( e^{2rA\sqrt{E_b} / N_0} - e^{-2rA\sqrt{E_b} / N_0} \right) p(A) dA > 0 \right\}
\]

\[
= \left\{ r : r > 0 \right\}
\]
The error probability is

\[
P_b = \int_0^\infty \left( \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \frac{(r + A \sqrt{E_b})^2}{N_0} \, dr \right) p(A) \, dA
\]

\[
= \int_0^\infty \left( \int_0^\infty e^{-\frac{(r + A \sqrt{E_b})^2}{N_0}} \, dr \right) p(A) \, dA
\]

\[
= \int_0^\infty \left( P \left[ N(-A \sqrt{E_b}, \frac{N_0}{2}) > 0 \right] \right) p(A) \, dA
\]

\[
= \int_0^\infty \left( P \left[ N(0,1) > \frac{A \sqrt{E_b}}{\sqrt{N_0/2}} \right] \right) p(A) \, dA
\]

\[
= \int_0^\infty Q\left( A \sqrt{2E_b / N_0} \right) p(A) \, dA
\]

\[
= E\left[ Q\left( A \sqrt{2E_b / N_0} \right) \right]
\]
For carrier modulated signals, \( \{s_m(t)\} \) are bandpass with lowpass equivalents \( s_{ml}(t) \)

\[
s_m(t) = \text{Re} \left[ s_{ml}(t)e^{j2\pi f_c t} \right]
\]

In AWGN channel,

\[
r(t) = s_m(t - t_d) + n(t)
\]

\( t_d \): random time asynchronism between tx and rx

\[
r(t) = \text{Re} \left[ s_{ml}(t - t_d)e^{j2\pi f_c (t-t_d)} \right] + n(t)
= \text{Re} \left[ s_{ml}(t - t_d)e^{-j2\pi f_c t_d} e^{j2\pi f_c t} \right] + n(t)
\]

Lowpass equivalent of \( s_m(t - t_d) \) is \( s_{ml}(t - t_d)e^{-j2\pi f_c t_d} \)

In practice, \( t_d \ll T_S \Rightarrow s_{ml}(t - t_d) \approx s_{ml}(t) \)

The random phase shift \( \phi = 2\pi f_c t_d \) could be large since \( f_c \) is large

\( \Rightarrow \) Noncoherent Detection
In the noncoherent case,
\[ \text{Re} \left[ r_l(t)e^{j2\pi f_c t} \right] = \text{Re} \left[ (e^{j\phi} s_{ml}(t) + n_l(t))e^{j2\pi f_c t} \right] \]

The baseband channel model:
\[ r_l(t) = e^{j\phi} s_{ml}(t) + n_l(t) \]

Vector equivalent form:
\[ r_l = e^{j\phi} s_{ml} + n_l \]

The optimum noncoherent detection is
\[ \hat{m} = \arg \max_{1 \leq m \leq M} \frac{P_m}{2\pi} \int_0^{2\pi} p_{n_l}(r_l - e^{j\phi} s_{ml})d\phi \]

From (4.5-3)
\[ = \arg \max_{1 \leq m \leq M} \frac{P_m}{2\pi} \frac{1}{(4\pi N_0)^N} \int_0^{2\pi} e^{-\frac{\|r_l - e^{j\phi} s_{ml}\|^2}{(4N_0)}} d\phi \]

\[ n_l \sim N(0_{N \times 1}, 2N_0 I_N) \]
Chaper 4.5-1 Noncoherent Detection of Carrier Modulated Signals

\[ \hat{m} = \arg \max_{1 \leq m \leq M} \frac{P_m}{2\pi} \left( \frac{1}{4\pi N_0} \right)^N \int_0^{2\pi} \left| r_i - e^{j\phi} s_{ml} \right|^2 \frac{d\phi}{(4N_0)^{\frac{1}{2}}}. \]

\[ = \arg \max_{1 \leq m \leq M} \frac{P_m}{2\pi} \left( \frac{1}{4\pi N_0} \right)^N \int_0^{2\pi} \left| e^{j\phi} \left( r_i \cdot s_{ml} \right) \right| \frac{d\phi}{(4N_0)^{\frac{1}{2}}}. \]

\[ = \arg \max_{1 \leq m \leq M} \frac{P_m}{2\pi} \left( \frac{1}{4\pi N_0} \right)^N \int_0^{2\pi} \left| e^{-j(\phi - \theta)} \right| \frac{d\phi}{(4N_0)^{\frac{1}{2}}}. \]

\[ = \arg \max_{1 \leq m \leq M} \frac{P_m}{2\pi} e^{- \frac{E_m}{2N_0}} \int_0^{2\pi} \left| e^{-j(\phi - \theta)} \right| \frac{d\phi}{(4N_0)^{\frac{1}{2}}}. \]

\[ = \arg \max_{1 \leq m \leq M} \frac{P_m}{2\pi} \left( \frac{1}{4\pi N_0} \right)^N \int_0^{2\pi} e^{-j(\phi - \theta)} \frac{d\phi}{(4N_0)^{\frac{1}{2}}}. \]

\[ s_m = s_{ml} \cos 2\pi f_c t \]

\[ E_m = \int \left| s_{ml} \right|^2 \cos^2 2\pi f_c t \ dt = \frac{\left| s_{ml} \right|^2}{2} \]

\[ \left| r_i - e^{j\phi} s_{ml} \right|^2 = \left| r_i \right|^2 - 2 \Re \{ r_i \cdot e^{j\phi} s_{ml} \} + \left| e^{j\phi} s_{ml} \right|^2 = \left| r_i \right|^2 - 2 \Re \{ r_i \cdot e^{j\phi} s_{ml} \} + 2E_m \]

\[ 
\begin{align*}
    r_i \cdot e^{j\phi} s_{ml} &= \left( e^{j\phi} s_{ml} \right)^H \cdot r = e^{-j\phi} (s_{ml})^H \cdot r \\
    &= e^{-j\phi} r \cdot s_{ml}
\end{align*}
\]

\[ r_i \cdot s_{ml} = \left| r_i \cdot s_{ml} \right| e^{i\theta} \]

\[ \theta : \text{phase of } r_i \cdot s_{ml} \]

\[ I_0(x) \text{ is modified Bessel function of the 1st kind and order zero} \]

\[ I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos \phi} d\phi \]
If signals are *equiprobable* and have *equal energy*,

\[
\hat{m} = \arg \max_{1 \leq m \leq M} I_0 \left( \frac{|r_l \cdot s_{ml}|}{2N_0} \right)
\]

\[= \arg \max_{1 \leq m \leq M} |r_l \cdot s_{ml}|\]

\[= \arg \max_{1 \leq m \leq M} \left| \int_{-\infty}^{\infty} r_l(t)s_{ml}^*(t)dt \right|\]
One can compare the digital modulation methods on the basis of the SNR required to achieve a specified probability of error.

However, such a comparison would not be very meaningful, unless it were made on the basis of some constraint, such as a fixed data rate of transmission or, on the basis of a fixed bandwidth.

For multiphase signals, the channel bandwidth required is simply the bandwidth of the equivalent low-pass signal pulse $g(t)$ with duration $T$ and bandwidth $W$, which is approximately equal to the reciprocal of $T$.

Since $T = k/R = (\log_2 M)/R$, it follows that $W = \frac{R}{\log_2 M}$.
4.6 Comparison of Digital Modulation Methods

- As $M$ is increased, the channel bandwidth required, when the bit rate $R$ is fixed, decreases. The *bandwidth efficiency* is measured by the **bit rate to bandwidth ratio**, which is
  \[ \frac{R}{W} = \log_2 M \]

- The bandwidth-efficient method for transmitting PAM is **single-sideband**. The channel bandwidth required to transmit the signal is approximately equal to $1/2T$ and,
  \[ \frac{R}{W} = 2 \log_2 M \]
  this is a factor of 2 better than PSK.

- For QAM, we have two orthogonal carriers, with each carrier having a PAM signal.
Thus, we double the rate relative to PAM. However, the QAM signal must be transmitted via double-sideband. Consequently, QAM and PAM have the same bandwidth efficiency when the bandwidth is referenced to the band-pass signal.

As for orthogonal signals, if the $M = 2^k$ orthogonal signals are constructed by means of orthogonal carriers with minimum frequency separation of $1/2T$, the bandwidth required for transmission of $k = \log_2 M$ information bits is

$$W = \frac{M}{2T} = \frac{M}{2(k/R)} = \frac{M}{2 \log_2 M} R$$

In the case, the bandwidth increases as $M$ increases.

In the case of biorthogonal signals, the required bandwidth is one-half of that for orthogonal signals.
A compact and meaningful comparison of modulation methods is one based on the normalized data rate $\frac{R}{W}$ (bits per second per hertz of bandwidth) versus the SNR per bit ($\frac{\varepsilon_b}{N_0}$) required to achieve a given error probability.

In the case of PAM, QAM, and PSK, increasing $M$ results in a higher bit-to-bandwidth ratio $\frac{R}{W}$. 
However, the cost of achieving the higher data rate is an increase in the SNR per bit.

Consequently, these modulation methods are appropriate for communication channels that are bandwidth limited, where we desire a $R/W > 1$ and where there is sufficiently high SNR to support increases in $M$.

- Telephone channels and digital microwave ratio channels are examples of such band-limited channels.

In contrast, $M$-ary orthogonal signals yield a $R/W \leq 1$. As $M$ increases, $R/W$ decreases due to an increase in the required channel bandwidth.

The SNR per bit required to achieve a given error probability decreases as $M$ increases.
Consequently, $M$-ary orthogonal signals are appropriate for power-limited channels that have sufficiently large bandwidth to accommodate a large number of signals.

As $M \rightarrow \infty$, the error probability can be made as small as desired, provided that $\text{SNR} > 0.693$ (-1.6dB). This is the minimum SNR per bit required to achieve reliable transmission in the limit as the channel bandwidth $W \rightarrow \infty$ and the corresponding $R/W \rightarrow 0$.

The figure above also shown the normalized capacity of the band-limited AWGN channel, which is due to Shannon (1948).

The ratio $C/W$, where $C (=R)$ is the capacity in bits/s, represents the highest achievable bit rate-to-bandwidth ratio on this channel.

Hence, it serves the upper bound on the bandwidth efficiency of any type of modulation.