Chapter 6
Carrier and Symbol Synchronization
Table of Contents

6.1 Signal Parameter Estimation
  6.1.1 The Likelihood Function
  6.1.2 Carrier Recovery and Symbol Synchronization in Signal Demodulation

6.2 Carrier Phase Estimation
  6.2.1 Maximum-Likelihood Carrier Phase Estimation
  6.2.2 The Phase-Locked Loop
  6.2.3 Effect of Additive Noise on the Phase Estimate
  6.2.4 Decision-Directed Loops
  6.2.5 Non-Decision-Directed Loops

6.3 Symbol Timing Estimation
  6.3.1 Maximum-Likelihood Timing Estimation
  6.3.2 Non-Decision-Directed Timing Estimation
In a digital communication system, the output of the demodulator must be sampled periodically, once per symbol interval, in order to recover the transmitted information.

Since the propagation delay from the transmitter to the receiver is generally unknown at the receiver, symbol timing must be derived from the received signal in order to synchronously sample the output of the demodulator.

The propagation delay in the transmitted signal also results in a carrier offset, which must be estimated at the receiver if the detector is phase-coherent.

*Symbol synchronization* is required in every digital communication system which transmits information synchronously.

*Carrier recovery* is required if the signal is detected coherently.
6.1 Signal Parameter Estimation

- We assume that the channel delays the signals transmitted through it and corrupts them by the addition of Gaussian noise.

- The received signal may be expressed as

\[ r(t) = s(t - \tau) + n(t) \]

where

\[ s(t) = \text{Re}\left[ s_l(t) e^{j2\pi f_c t} \right] \]

\( \tau \): propagation delay

\( s_l(t) \): the equivalent low-pass signal

- The received signal may be expressed as:

\[ r(t) = \text{Re}\left\{ [s_l(t - \tau) e^{j\phi} + z(t)] e^{j2\pi f_c t} \right\} \]

where the carrier phase \( \phi \), due to the propagation delay \( \tau \), is \( \phi = -2\pi f_c \tau \).
6.1 Signal Parameter Estimation

- It may appear that there is only one signal parameter to be estimated, the propagation delay, since one can determine $\phi$ from knowledge of $f_c$ and $\tau$. However, the received carrier phase is not only dependent on the time delay $\tau$ because:
  - The oscillator that generates the carrier signal for demodulation at the receiver is generally not synchronous in phase with that at the transmitter.
  - The two oscillators may be drifting slowly with time.
- The precision to which one must synchronize in time for the purpose of demodulating the received signal depends on the symbol interval $T$. Usually, the estimation error in estimating $\tau$ must be a relatively small fraction of $T$.
  - $\pm1$ percent of $T$ is adequate for practical applications. However, this level of precision is generally inadequate for estimating the carrier phase since $f_c$ is generally large.
6.1 Signal Parameter Estimation

- In effect, we must estimate both parameters $\tau$ and $\phi$ in order to demodulate and coherently detect the received signal.
- Hence, we may express the received signal as
  \[ r(t) = s(t; \phi, \tau) + n(t) \]
  where $\phi$ and $\tau$ represent the signal parameters to be estimated.
- To simplify the notation, we let $\psi$ denote the parameter vector $\{\phi, \tau\}$, so that $s(t; \phi, \tau)$ is simply denoted by $s(t; \psi)$.
- There are two criteria that are widely applied to signal parameter estimation: the maximum-likelihood (ML) criterion and the maximum a posteriori probability (MAP) criterion.
  - In the MAP criterion, $\psi$ is modeled as random and characterized by an a priori probability density function $p(\psi)$.
  - In the ML criterion, $\psi$ is treated as deterministic but unknown.
6.1 Signal Parameter Estimation

By performing an orthonormal expansion of $r(t)$ using $N$ orthonormal functions $\{f_n(t)\}$, we may represent $r(t)$ by the vector of coefficients $[r_1 \ r_2 \ \cdots \ r_N] \equiv \mathbf{r}$.

The joint PDF of the random variables $[r_1 \ r_2 \ \cdots \ r_N]$ in the expansion can be expressed as $p(\mathbf{r} \mid \psi)$.

- The ML estimate of $\psi$ is the value that maximizes $p(\mathbf{r} \mid \psi)$.
- The MAP estimate is the value of $\psi$ that maximizes the a posteriori probability density function

$$p(\psi \mid \mathbf{r}) = \frac{p(\mathbf{r} \mid \psi) p(\psi)}{p(\mathbf{r})}$$

If there is no prior knowledge of the parameter vector $\psi$, we may assume that $p(\psi)$ is uniform (constant) over the range of values of the parameters.
In such a case, the value of $\psi$ that maximizes $p(r \mid \psi)$ also maximizes $p(\psi \mid r)$. Therefore, the MAP and ML estimates are identical.

In our treatment of parameter estimation given below, we view the parameters $\phi$ and $\tau$ as unknown, but deterministic. Hence, we adopt the ML criterion for estimating them.

In the ML estimation of signal parameters, we require that the receiver extract the estimate by observing the received signal over a time interval $T_0 \geq T$, which is called the *observation interval*.

Estimates obtained from a single observation interval are sometimes called *one-shot estimates*.

In practice, the estimation is performed on a continuous basis by using tracking loops (either analog or digital) that continuously update the estimates.
6.1.1 The Likelihood Function

Since the additive noise $n(t)$ is white and zero-mean Gaussian, the joint PDF $p(r|\psi)$ may be expressed as

$$p(r|\psi) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^N \exp\left\{ -\sum_{n=1}^{N} \frac{[r_n - s_n(\psi)]^2}{2\sigma^2} \right\} \quad \text{--- (A)}$$

where

$$r_n = \int_{T_0} r(t) f_n(t) dt \quad s_n(\psi) = \int_{T_0} s(t; \psi) f_n(t) dt \quad \text{--- (B)}$$

where $T_0$ represents the integration interval in the expansion of $r(t)$ and $s(t; \psi)$.

By substituting from Equation (B) into Equation (A):

$$\lim_{N \to \infty} \frac{1}{2\sigma^2} \sum_{n=1}^{N} [r_n - s_n(\psi)]^2 = \frac{1}{N_0} \int_{T_0} \left[ r(t) - s(t; \psi) \right]^2 dt$$
6.1.1 The Likelihood Function

Now, the maximization of $p(r|\psi)$ with respect to the signal parameters $\psi$ is equivalent to the maximization of the likelihood function.

$$\Lambda(\psi) = \exp \left\{ -\frac{1}{N_0} \int_{T_0} [r(t) - s(t; \psi)]^2 dt \right\} \quad (6.1-8)$$

Below, we shall consider signal parameter estimation from the viewpoint of maximizing $\Lambda(\psi)$. 
6.1.2 Carrier Recovery and Symbol Synchronization in Signal Demodulation

- **Binary PSK signal demodulator and detector**
  - The carrier phase estimate is used in generating the reference signal $g(t)\cos(2\pi f_c t + \hat{\phi})$ for the correlator.
  - The symbol synchronizer controls the sampler and the output of the signal pulse generator.
  - If the signal pulse is rectangular, then the signal generator can be eliminated.
6.1.2 Carrier Recovery and Symbol Synchronization in Signal Demodulation

- \( M \)-ary PSK signal demodulator and detector:
  - Two correlators (or matched filters) are required: \( g(t)\cos(2\pi f_c t + \hat{\phi}) \) and \( g(t)\sin(2\pi f_c t + \hat{\phi}) \), where \( \hat{\phi} \) is the carrier phase estimate.
  - The detector is a phase detector, which compares the received signal phases with the possible transmitted signal phases.
6.1.2 Carrier Recovery and Symbol Synchronization in Signal Demodulation

- *M*-ary PAM signal demodulator and detector:
M-ary PAM signal demodulator and detector:

- A single correlator is required, and the detector is an amplitude detector, which compares the received signal amplitude with the possible transmitted signal amplitudes.
- The purpose of an automatic gain control (AGC) is to eliminate channel gain variations, which would affect the amplitude detector.
- The AGC has a relatively long time constant, so that it does not respond to the signal amplitude variations that occur on a symbol-by-symbol basis.
- The AGC maintains a fixed average (signal plus noise) power at its output.
6.1.2 Carrier Recovery and Symbol Synchronization in Signal Demodulation

QAM signal demodulator and detector:
6.1.2 Carrier Recovery and Symbol Synchronization in Signal Demodulation

- QAM signal demodulator and detector
  - An AGC is required to maintain a constant average power signal at the input to the demodulator.
  - The demodulator is similar to a PSK demodulator, in that both generate in-phase and quadrature signal samples \((X, Y)\) for the detector.
  - The detector computes the Euclidean distance between the received noise-corrupted signal point and the \(M\) possible transmitted points, and selects the signal closest to the received point.
6.2 Carrier Phase Estimation

Two basic approaches for dealing with carrier synchronization at the receiver:

- One is to multiplex, usually in frequency, a special signal, called a *pilot signal*, that allows the receiver to extract and to synchronize its local oscillator to the carrier frequency and phase of the received signal.

- When an unmodulated carrier component is transmitted along with the information-bearing signal, the receiver employs a *phase-locked loop (PLL)* to acquire and track the carrier component.

- The PLL is designed to have a narrow bandwidth so that it is not significantly affected by the presence of frequency components from the information-bearing signal.
The second approach, which appears to be more prevalent in practice, is to derive the carrier phase estimate directly from the modulated signal.

- This approach has the distinct advantage that the total transmitter power is allocated to the transmission of the information-bearing signal.
- In our treatment of carrier recovery, we confine our attention to the second approach; hence, we assume that the signal is transmitted via suppressed carrier.
6.2 Carrier Phase Estimation

- Consider the effect of a carrier phase error on the demodulation of a double-sideband, suppressed carrier (DSB/SC) signal:
  - Suppose we have an amplitude-modulated signal:
    \[ s(t) = A(t) \cos(2\pi f_c t + \phi) \]
  - Demodulate the signal by multiplying \( s(t) \) with the carrier reference:
    \[ c(t) = \cos(2\pi f_c t + \hat{\phi}) \]
    we obtain
    \[ c(t)s(t) = \frac{1}{2} A(t) \cos(\phi - \hat{\phi}) + \frac{1}{2} A(t) \cos(4\pi f_c t + \phi + \hat{\phi}) \]
  - Passing the product signal \( c(t)s(t) \) though a low-pass filter:
    \[ y(t) = \frac{1}{2} A(t) \cos(\phi - \hat{\phi}) \]
6.2 Carrier Phase Estimation

- The effect of the phase error $\phi - \hat{\phi}$ is to reduce the signal level in voltage by a factor $\cos(\phi - \hat{\phi})$ and in power by a factor $\cos^2(\phi - \hat{\phi})$.

- Hence, a phase error of 10° results in a signal power loss of 0.13 dB, and a phase error of 30° results in a signal power loss of 1.25 dB in an amplitude-modulated signal.

- The effect of carrier phase errors in QAM and multiphase PSK is much more severe.

- The QAM and $M$-PSK signals may be represented as:

$$s(t) = A(t)\cos(2\pi f_c t + \phi) - B(t)\sin(2\pi f_c t + \phi)$$

- Demodulated by the two quadrature carriers:

$$c_c(t) = \cos(2\pi f_c t + \hat{\phi}) \quad c_s(t) = -\sin(2\pi f_c t + \hat{\phi})$$
6.2 Carrier Phase Estimation

- Multiplication of $s(t)$ with $c_c(t)$ and $c_s(t)$ followed by low-pass filtering yield the in-phase and quadrature component:
  \[ y_I(t) = \frac{1}{2} A(t) \cos(\phi - \hat{\phi}) - \frac{1}{2} B(t) \sin(\phi - \hat{\phi}) \]
  \[ y_Q(t) = \frac{1}{2} B(t) \cos(\phi - \hat{\phi}) + \frac{1}{2} A(t) \sin(\phi - \hat{\phi}) \]

- The phase error in the demodulation of QAM and $M$-PSK signals has a much more severe effect than in the demodulation of a PAM signal:
  - There is a reduction in the power of the desired signal component by a factor of $\cos^2(\phi - \hat{\phi})$.
  - There is also crosstalk interference from the in-phase and quadrature components. Since the average power levels of $A(t)$ and $B(t)$ are similar, a small phase error causes a large degradation in performance.
  - Hence, the phase accuracy requirements for QAM and multiphase coherent PSK are much higher than for DSB/SC PAM.
We derive the maximum-likelihood carrier phase estimate.

Assuming that the delay $\tau$ is known, and, we set $\tau = 0$.

The function to be maximized is the likelihood function (6.1-8):

$$\Lambda (\psi) = \exp \left\{ -\frac{1}{N_0} \int_{T_0} \left[ r(t) - s(t; \psi) \right]^2 dt \right\}$$

With $\phi$ substituted for $\psi$, the function becomes

$$\Lambda (\phi) = \exp \left\{ -\frac{1}{N_0} \int_{T_0} \left[ r(t) - s(t; \phi) \right]^2 dt \right\}$$

$$= \exp \left\{ -\frac{1}{N_0} \int_{T_0} r^2(t) dt + \frac{2}{N_0} \int_{T_0} r(t) s(t; \phi) dt - \frac{1}{N_0} \int_{T_0} s^2(t; \phi) dt \right\}$$

A constant, equal to the signal energy over the observation interval $T_0$ for any value of $\phi$. Independent of $\phi$
Only the second term of the exponential factor involves the cross correlation of the received signal $r(t)$ with $s(t; \phi)$, depends on the choose of $\phi$.

Therefore, the likelihood function $\Lambda(\phi)$ may be expressed as

$$
\Lambda(\phi) = C \exp \left[ \frac{2}{N_0} \int_{T_0} r(t) s(t; \phi) \, dt \right]
$$

where $C$ is a constant independent of $\phi$.

The ML estimate $\hat{\phi}_{ML}$ is the value of $\phi$ that maximizes $\Lambda(\phi)$. Equivalently, it also maximizes the logarithm of $\Lambda(\phi)$, i.e., the log-likelihood function:

$$
\Lambda_L(\phi) = \frac{2}{N_0} \int_{T_0} r(t) s(t; \phi) \, dt
$$
6.2.1 Maximum-Likelihood Carrier Phase Estimation

Example 6.2-1: Transmission of the unmodulated carrier $A\cos 2\pi f_c t$:

- The received signal $r(t)$ is
  $$ r(t) = A \cos (2\pi f_c t + \phi) + n(t) $$
  where $\phi$ is the unknown phase.
- We seek the value of $\phi$, say $\hat{\phi}_{ML}$, that maximize
  $$ \Lambda_L(\phi) = \frac{2A}{N_0} \int_{T_0}^T r(t) \cos (2\pi f_c t + \phi) \, dt $$
- A necessary condition for a maximum is
  $$ \frac{d\Lambda_L(\phi)}{d\phi} = 0 $$
  $$ \int_{T_0}^T r(t) \sin (2\pi f_c t + \hat{\phi}_{ML}) \, dt = 0 $$
  (A)
  $$ \hat{\phi}_{ML} = -\tan^{-1} \left[ \frac{\int_{T_0}^T r(t) \sin 2\pi f_c t \, dt}{\int_{T_0}^T r(t) \cos 2\pi f_c t \, dt} \right] $$
  (B)
Example 6.2-1: (cont.)

- Equation (A) implies the use of a loop to extract the estimate:

The loop filter is an integrator whose bandwidth is proportional to the reciprocal of the integration interval $T_0$. 
Example 6.2-1: (cont.)

Equation (B) implies an implementation that uses quadrature carriers to cross-correlate with $r(t)$.
6.2.2 The Phase-Locked Loop

The PLL basically consists of a multiplier, a loop filter, and a voltage-controlled oscillator (VCO):

Assuming that the input to the PLL is the sinusoid \( x_c(t) = A_c \cos(2\pi f_c t + \phi) \) and the output of the VCO is \( e_0(t) = -A_v \sin(2\pi f_c t + \phi) \), where \( \phi \) represents the estimate of \( \phi \), the product of two signals is:

\[
e_d(t) = x_c(t) e_0(t) = -A_c \cos(2\pi f_c t + \phi) A_v \sin(2\pi f_c t + \hat{\phi})
\]

\[
= \frac{1}{2} A_c A_v \sin(\phi - \hat{\phi}) - \frac{1}{2} A_c A_v \sin(4\pi f_c t + \phi + \hat{\phi})
\]
6.2.2 The Phase-Locked Loop

- The loop filter is a low-pass filter that responds only to the low-frequency component \(0.5A_c A_v \sin(\phi - \hat{\phi})\) and removes the component at \(2f_c\).
- The output of the loop filter provides the control voltage \(e_v(t)\) for the VCO.
- The VCO is a sinusoidal signal generator with an instantaneous phase given by

\[
2\pi f_c t + \hat{\phi}(t) = 2\pi f_c t + K_v \int_{-\infty}^{t} e_v(\tau)d\tau
\]

where \(K_v\) is a gain constant in rad/s/V.

\[
\hat{\phi}(t) = K_v \int_{-\infty}^{t} e_v(\tau)d\tau \quad \text{or} \quad \frac{d\hat{\phi}}{dt} = K_v e_v(t)
\]
6.2.2 The Phase-Locked Loop

By neglecting the double-frequency term resulting from the multiplication of the input signal with the output of the VCO, the phase detector output is:

\[ e_d(\psi) = K_d \sin \psi \]

where \( \psi = \phi - \hat{\phi} \) is the phase error and \( K_d \) is a proportionality constant.

In normal operation, when the loop is tracking the phase of the incoming carrier, the phase error \( \phi - \hat{\phi} \) is small. As a result,

\[ \sin(\phi - \hat{\phi}) \approx \phi - \hat{\phi} \]

With the assumption that \( |\psi|<<1 \), the PLL becomes linear.
6.2.2 The Phase-Locked Loop

- The equations describing loop operation is conveniently obtained by using Laplace transform notation.
- A loop model using Laplace-transformed quantities and assuming linear operation is shown in the following figure:
6.2.2 The Phase-Locked Loop

- The Laplace-transformed loop equations are:
  \[ E_d(s) = K_d \left[ \Phi(s) - \Theta(s) \right] = K_d \Psi(s) \]
  \[ E_v(s) = F(s) E_d(s) \]
  \[ \Theta(s) = \frac{K_v E_v(s)}{s} \]

- The closed-loop transfer function:
  \[ H(s) \triangleq \frac{\Theta(s)}{\Phi(s)} = \frac{K_v D_d F(s)}{s + K_v D_d F(s)} \triangleq \frac{K F(s) / s}{1 + K F(s) / s} \]

- The phase error transfer function:
  \[ H_e(s) \triangleq \frac{\Phi(s) - \Theta(s)}{\Phi(s)} = \frac{\Psi(s)}{\Phi(s)} = 1 - \frac{\Theta(s)}{\Phi(s)} = 1 - H(s) = \frac{s}{s + K_v K_d F(s)} \]
6.2.2 The Phase-Locked Loop

The VCO control-voltage/input-phase transfer function:

\[ H_v(s) = \frac{E_v(s)}{\Phi(s)} = \frac{sH(s)}{K_v} = \frac{K_d sF(s)}{s + K_v K_d F(s)} \]

It is convenient to write the closed-loop transfer function in terms of the open-loop transfer function, which is defined as:

\[ G_{op}(s) \triangleq \frac{K_v K_d F(s)}{s} \implies H(s) = \frac{G_{op}(s)}{1 + G_{op}(s)} \]

\( K = K_v K_d \) is the open-loop dc gain.

By appropriate choice of \( F(s) \), any order closed-loop transfer function can be obtained.

For second-order passive loops, the transfer function is:

\[ F(s) = \frac{1 + \tau_2 s}{1 - \tau_1 s} \implies H(s) = \frac{1 + \tau_2 s}{1 + \left( \tau_2 + 1/K \right)s + \left( \tau_1 / K \right)s^2} \]
6.2.2 The Phase-Locked Loop

- Second-order phase-locked-loop filters

(a) Passive

(b) Active
### 6.2.2 The Phase-Locked Loop

Transfer functions and parameters for first- and second-order phase-locked loops

#### TABLE A-1. Transfer Functions and Parameters for First- and Second-Order Phase-Locked Loops

<table>
<thead>
<tr>
<th>Loop Filter, $F(s)$</th>
<th>Natural Frequency, $\omega_n$ (rad/s)</th>
<th>Damping Factor $\zeta$</th>
<th>Closed-Loop Transfer Function, $H(s)$</th>
<th>Error Transfer Function, $1 - H(s)$</th>
<th>Single-Sided Noise/Equivalent Bandwidth $b^c$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (first order)</td>
<td>$K$</td>
<td>—</td>
<td>$\frac{K}{s + K}$</td>
<td>$\frac{s}{s + K}$</td>
<td>$K$</td>
</tr>
<tr>
<td>$s\tau_2 + 1$</td>
<td>$\sqrt{\frac{K}{\tau_1}}$</td>
<td>$\frac{\omega_n}{2}(\tau_2 + K^{-1})$</td>
<td>$\frac{(2\zeta\omega_n - \omega_n^2/K)s + \omega_n^2}{D(s)}$</td>
<td>$\frac{s^2 + \omega_n^2 s/K}{D(s)}$</td>
<td>$\frac{1}{4} K_{\tau_1}^2 + K/\tau_1$</td>
</tr>
<tr>
<td>$s\tau_2 + 1$</td>
<td>$\sqrt{\frac{K}{\tau_1}}$</td>
<td>$\frac{\tau_2\omega_n}{2}$</td>
<td>$\frac{2\zeta\omega_n s + \omega_n^2}{D(s)}$</td>
<td>$\frac{s^2}{D(s)}$</td>
<td>$\frac{1}{2} \omega_n \left( \zeta + \frac{1}{4\zeta} \right)$</td>
</tr>
<tr>
<td>$\frac{1}{s\tau_1}$</td>
<td>$\sqrt{\frac{K}{\tau}}$</td>
<td>$\frac{1}{2\sqrt{K\tau}}$</td>
<td>$\frac{\omega_n^2}{D(s)}$</td>
<td>$\frac{s^2 + 2\zeta\omega_n}{D(s)}$</td>
<td>$K$</td>
</tr>
</tbody>
</table>

$^a K = K_1 K_d$.

$^b$ The noise equivalent bandwidth of a filter with transfer function $H(f)$ and maximum gain $H_0$ is given by $B_N = (1/H_0) \int_0^\infty |H(f)|^2 df$.

$^c$ For a second-order loop with $\zeta = 0.5$, $B_L = 0.5\omega_n$; with $\zeta = 1/\sqrt{2}$, $B_L = 0.53\omega_n$. $B_L$ is the single-sided noise equivalent bandwidth in hertz, and the dimensions of $\omega_n$ are rad/s.
6.2.2 The Phase-Locked Loop

- Hence, the closed-loop system for the linearized PLL is second-order.
- It is customary to express the denominator of $H(s)$ in the standard form:

$$D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

where $\xi$: loop damping factor

$\omega_n$: natural frequency of the loop

$\omega_n = \sqrt{K/\tau_1}$ and $\xi = \omega_n (\tau_2 + 1/K)/2$

- The closed-loop transfer function becomes:

$$H(s) = \frac{2\zeta\omega_n - \omega_n^2 / K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$
6.2.2 The Phase-Locked Loop

The frequency response of a second-order loop (with $\tau_1 \gg 1$)

- $\zeta = 1 \Rightarrow$ critically damped loop response.
- $\zeta < 1 \Rightarrow$ underdamped response.
- $\zeta > 1 \Rightarrow$ overdamped response.
6.2.2 The Phase-Locked Loop

- In practice, the selection of the bandwidth of the PLL involves a trade-off between speed of response and noise in the phase estimate.
- On the one hand, it is desirable to select the bandwidth of the loop to be sufficiently wide to track any time variations in the phase of the received carrier.
- On the other hand, a wideband PLL allows more noise to pass into the loop, which corrupts the phase estimate.

6.2.3 Effect of Additive Noise on the Phase Estimate

- Assume that the noise at the input to the PLL is narrowband.
- We further assume that the PLL is tracking a sinusoidal signal of the form:
  \[ s(t) = A_c \cos\left(2\pi f_c t + \phi(t)\right) \]
- The signal is corrupted by the additive narrowband noise:
  \[ n(t) = x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t \]
- The in-phase and quadrature components of the noise are assumed to be statistically independent, stationary Gaussian noise processes with (two-sided) power spectral density \( \frac{1}{2} N_0 \) W/Hz.
  \[ n(t) = n_c(t)\cos\left(2\pi f_c t + \phi(t)\right) - n_s(t)\sin\left(2\pi f_c t + \phi(t)\right) \]
  where
  \[ n_c(t) = x(t)\cos \phi(t) + y(t)\sin \phi(t) \]
  \[ n_s(t) = -x(t)\sin \phi(t) + y(t)\cos \phi(t) \]
We note that \( n_c(t) + jn_s(t) = \left[ x(t) + jy(t) \right] e^{-j\phi(t)} \)

If \( s(t) + n(t) \) is multiplied by the output of the VCO and the double-frequency terms are neglected the input to the loop filter is the noise-corrupted signal

\[
e(t) = A_c \sin \Delta \phi + n_c(t) \sin \Delta \phi - n_s(t) \cos \Delta \phi
\]

\[
= A_c \sin \Delta \phi + n_1
\]

where \( \Delta \phi = \hat{\phi} - \phi \) is the phase error.
When the power $P_c = 0.5 A_c^2$ of the incoming signal is much larger than the noise power, we may linearize the PLL and, thus, easily determine the effect of the additive noise on the quality of the estimate $\hat{\phi}$.

The model for the linearized PLL with additive noise is:

The gain parameter $A_c$ may be normalized to unity. Thus:

$$n_2(t) = \frac{n_c(t)}{A_c} \sin \Delta \phi - \frac{n_s(t)}{A_c} \cos \Delta \phi$$
The noise term $n_2(t)$ is zero-mean Gaussian with a power spectral density $N_0/2A_c^2$.

Since the noise is additive at the input to the loop, the variance of the phase error $\Delta \phi$, which is also the variance of the VCO output phase, is:

$$\sigma^2_\phi = \frac{N_0}{2A_c^2} \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{A_c^2} \int_{0}^{\infty} |H(f)|^2 df \equiv \frac{N_0B_{eq}}{A_c^2}$$

where $B_{eq}$ is the (one-sided) equivalent noise bandwidth of the loop defined as:

$$B_{eq} = \frac{1}{G} \int_{0}^{\infty} |H(f)|^2 df$$

where $G=\max|H(f)|^2$. 

6.2.3 Effect of Additive Noise on the Phase Estimate
Note that $\sigma_\phi^2$ is simply the ratio of total noise power within the bandwidth of the PLL divided by the signal power.

Define the signal-to-noise ratio as:

$$\text{SNR} \equiv \gamma_L = 1/\sigma_\phi^2 = \frac{A_c^2}{N_0 B_{\text{eq}}}$$

The expression for the variance of the VCO phase error applies to the case where the SNR is sufficiently high that the linear model for the PLL applies.

An exact analysis based on the non-linear PLL is mathematically tractable when $G(s)=1$, which results in a first order loop. In this case the probability density function for the phase error has the form:

$$p(\Delta \phi) = \frac{\exp(\gamma_L \cos \Delta \phi)}{2\pi I_0(\gamma_L)}$$
6.2.3 Effect of Additive Noise on the Phase Estimate

- Comparison of VCO phase variance for exact and approximate (linear model) first-order PLL.
- Note that the variance for the linear model is close to the exact variance for $\gamma_L > 3$. 

![Graph showing variance of VCO phase estimate against $N_0B_{eq}/A_c^2$]
6.2.4 Decision-Directed Loops

- Up to this point, we consider carrier phase estimation when the carrier signal is unmodulated.

- We consider carrier phase recovery when the signal carries information \( \{I_n\} \). In this case, we can adopt one of two approaches: either we assume that \( \{I_n\} \) is known or we treat \( \{I_n\} \) as a random sequence and average over its statistics.

- In *decision-directed parameter estimation*, we assume that the information sequence over the observation interval has been estimated.

- Consider the decision-directed phase estimate for the class of linear modulation techniques for which the received equivalent low-pass signal may be expressed as:

\[
   r(t) = e^{-j\phi} \sum_{n} I_n g(t - nT) + z(t) = s(t) e^{-j\phi} + z(t)
\]
The likelihood function and corresponding log-likelihood function for the equivalent low-pass signal are (from 6.2-9 & 10):

\[ \Lambda(\phi) = C \exp \left\{ \text{Re} \left[ \frac{1}{N_0} \int_{T_0}^{T} r_l(t) s_l^*(t) e^{j\phi} dt \right] \right\} \]

\[ \Lambda_L(\phi) = \text{Re} \left\{ \frac{1}{N_0} \int_{T_0}^{T} r_l(t) s_l^*(t) dt \right\} e^{j\phi} \]

If we substitute for \( s_l(t) \) and assume that the observation interval \( T_0 = KT \), where \( K \) is a positive integer, we obtain:

\[ \Lambda_L(\phi) = \text{Re} \left\{ e^{j\phi} \frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* \int_{nT}^{(n+1)T} r_l(t) g^*(t-nT) dt \right\} = \text{Re} \left\{ e^{j\phi} \frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* y_n \right\} \]

\[ y_n = \int_{nT}^{(n+1)T} r_l(t) g^*(t-nT) dt \]
6.2.4 Decision-Directed Loops

- Differentiating the log-likelihood function with respect to $\phi$ and setting the derivative equal to zero:

$$\Lambda_L(\phi) = \text{Re}\left(\frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* y_n\right) \cos \phi - \text{Im}\left(\frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* y_n\right) \sin \phi = 0$$

$$\hat{\phi}_{ML} = -\tan^{-1}\left[\frac{\text{Im}\left(\sum_{n=0}^{K-1} I_n^* y_n\right)}{\text{Re}\left(\sum_{n=0}^{K-1} I_n^* y_n\right)}\right] \quad (6.2-38)$$

- $\hat{\phi}_{ML}$ is the decision-directed (or decision-feedback) carrier phase estimate.

- It can be shown that the mean value of $\hat{\phi}_{ML}$ is $\phi$. -- unbias.
6.2.4 Decision-Directed Loops

Double-sideband PAM signal receiver with decision-directed carrier phase estimation

Diagram:
- Received signal
- Multiplication
- Integration
- Sampler
- Amplitude detector
- Phase estimator
- Carrier generator
- Signal pulse generator
- Time sync.

Mathematical expression:
\[ \cos(2\pi f_c t + \hat{\phi}_{ML}) \]
Another implementation of the PAM receiver that employs a decision-feedback PLL (DFPLL) for carrier phase estimation is shown below:

Carrier recovery with a decision-feedback PLL.
The received double-sideband PAM is given by
\[ A(t) \cos(2 \pi f_c t + \phi), \] where \( A(t) = A_m g(t) \) and \( g(t) \) is assumed to be a rectangular pulse of duration \( T \).

The received signal is multiplied by the quadrature carriers \( c_c(t) \) and \( c_s(t) \). The product signal:

\[
r(t) \cos \left( 2\pi f_c t + \hat{\phi} \right) = \frac{1}{2} \left[ A(t) + n_c(t) \right] \cos \Delta \phi - \frac{1}{2} n_s(t) \sin \Delta \phi + \text{double-frequency terms}
\]

is used to recover the information carried by \( A(t) \).

The detector makes a decision on the symbol that is received every \( T \) seconds.

In the absence of decision errors, it reconstructs \( A(t) \) free of any noise.
The reconstructed signal is used to multiply the product of the second quadrature multiplier, which has been delayed by $T$ seconds to allow the demodulator to reach a decision.

The input to the loop filter in the absence of decision errors is the error signal:

$$e(t) = \frac{1}{2} A(t) \left\{ A(t) + n_c(t) \right\} \sin \Delta \phi - n_s(t) \cos \Delta \phi$$

+ double-frequency derms

$$= \frac{1}{2} A^2(t) \sin \Delta \phi + \frac{1}{2} A(t) \left[ n_c(t) \sin \Delta \phi - n_s(t) \cos \Delta \phi \right]$$

+ double-frequency derms

The loop filter rejects the double-frequency term.

The desired component is $A^2(t) \sin \Delta \phi$, which contains the phase error for driving the loop.
The ML estimate in 6.2-38 is also appropriate for QAM.

\[
\hat{\phi}_{ML} = -\tan^{-1}\left[ \frac{\text{Im}\left(\sum_{n=0}^{K-1} I_n^* y_n\right)}{\text{Re}\left(\sum_{n=0}^{K-1} I_n^* y_n\right)} \right]
\]
6.2.4 Decision-Directed Loops

- Carrier recovery for $M$-ary PSK using a decision-feedback PLL
The received signal is demodulated to yield the phase estimate

\[ \hat{\theta}_m = \frac{2\pi}{M} (m - 1) \]

which, in the absence of noise, is the transmitted signal phase.

The two outputs of the quadrature multipliers are delayed by the symbol duration \( T \) and multiplied by \( \cos \theta_m \) and \( \sin \theta_m \):

\[
r(t) \cos \left( 2\pi f_c t + \hat{\phi} \right) \sin \theta_m = \frac{1}{2} \left[ A \cos \theta_m + n_c(t) \right] \sin \theta_m \cos (\phi - \hat{\phi})
- \frac{1}{2} \left[ A \sin \theta_m + n_s(t) \right] \sin \theta_m \sin (\phi - \hat{\phi}) + \text{double-frequency terms}
\]

\[
r(t) \sin \left( 2\pi f_c t + \hat{\phi} \right) \cos \theta_m = -\frac{1}{2} \left[ A \cos \theta_m + n_c(t) \right] \cos \theta_m \sin (\phi - \hat{\phi})
- \frac{1}{2} \left[ A \sin \theta_m + n_s(t) \right] \cos \theta_m \cos (\phi - \hat{\phi}) + \text{double-frequency terms}
\]
6.2.4 Decision-Directed Loops

- The two signals are added to generate the error signal:
  \[ e(t) = -\frac{1}{2} A \sin(\phi - \hat{\phi}) + \frac{1}{2} n_c(t) \sin(\phi - \hat{\phi} - \theta_m) \]
  + \[ \frac{1}{2} n_s(t) \cos(\phi - \hat{\phi} - \theta_m) \] + double-frequency terms

- This error signal is the input to the loop filter that provides the control signal for the VCO.

- We observe that the two quadrature noise components in (6.2-42) appear as additive terms. There is no term involving a product of two noise components.

- This \( M \)-phase tracking loop has a phase ambiguity of \( 360^{\circ}/M \), necessitating the need to differentially encode the information sequence prior to transmission and differentially decode the received sequence after demodulation.
Instead of using a decision-directed scheme to obtain the phase estimate, we may treat the data as random variables and simply average $\Lambda(\phi)$ over these random variables prior to maximization.

In order to carry out this integration, we may use:

- The actual probability distribution function of the data, if it is known.
- Assume some probability distribution that might be a reasonable approximation to the true distribution.

The following example illustrates the first approach.
Example 6.2-2.

Suppose the real signal $s(t)$ carries binary modulation. Then, in a signal interval, we have:

$$s(t) = A \cos 2\pi f_c t, \quad 0 \leq t \leq T$$

where $A=\pm 1$ with equal probability. Clearly, the PDF of $A$ is given as:

$$p(A) = \frac{1}{2} \delta(A - 1) + \frac{1}{2} \delta(A + 1)$$

The likelihood function given by Equation 6.2-9 is conditional on a given value of $A$ and must be averaged over the two values.
Thus, we have
\[
\bar{\Lambda}(\phi) = \int_{-\infty}^{\infty} \Lambda(\phi) p(A) dA
\]
\[
= \frac{1}{2} \exp \left[ \frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right]
\]
\[
+ \frac{1}{2} \exp \left[ - \frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right]
\]
\[
= \cosh \left[ \frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right]
\]

The corresponding log-likelihood function is:
\[
\bar{\Lambda}_L(\phi) = \ln \cosh \left[ \frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) dt \right] \quad --- \text{(A)}
\]
If we differentiate the log-likelihood function and set the derivative equal to zero, we can obtain the ML estimate for the non-decision-directed estimate. Unfortunately, the relationship in Equation A is highly non-linear and, hence, an exact solution is difficult to obtain.

On the other hand, approximations are possible. In particular,

\[
\ln \cosh x = \begin{cases} 
\frac{1}{2} x^2 & (|x| \ll 1) \\
|x| & (|x| \gg 1)
\end{cases}
\]

With these approximations, the solution for \( \phi \) becomes tractable.
Example 6.2-3.

Consider the same signal as in Example 6.2-2, but now assume that the amplitude \( A \) is zero-mean Gaussian with unit variance.

\[
p(A) = \frac{1}{\sqrt{2\pi}} e^{-A^2/2}
\]

If we average \( \Lambda(\phi) \) over the assumed PDF of \( A \), we obtain the average likelihood in the form:

\[
\bar{\Lambda}(\phi) = C \exp \left\{ \frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) \, dt \right\}^2
\]

The corresponding log-likelihood is:

\[
\bar{\Lambda}_L(\phi) = \left[ \frac{2}{N_0} \int_0^T r(t) \cos(2\pi f_c t + \phi) \, dt \right]^2
\]

ML estimate of \( \phi \) is obtained by differentiating the above equation and setting the derivative to zero.
The log-likelihood function is quadratic under the Gaussian assumption and it is approximately quadratic (6.2-45) for small values of the cross correlation of $r(t)$ with $s(t; \phi)$.

In other words, if the cross correlation over a single interval is small, the Gaussian assumption for the distribution of the information symbols yields a good approximation to the log-likelihood function.

We may use the Gaussian approximation on all the symbols in the observation interval $T_0=KT$. Specifically, we assume that the $K$ information symbols are statistically independent and identically distributed.
By averaging the likelihood function $\Lambda(\phi)$ over the Gaussian PDF for each of the $K$ symbols in the interval $T_0=KT$, we obtain the result:

$$\overline{\Lambda}(\phi) = C \exp \left\{ \sum_{n=0}^{K-1} \frac{2}{N_0} \int_{nT}^{(n+1)T} r(t) \cos(2\pi f_c t + \phi) \, dt \right\}^2$$

If we take the logarithm, differentiate the resulting log-likelihood function, and set the derivative equal to zero, we obtain the condition for the $M$ estimate as:

$$\sum_{n=0}^{K-1} \int_{nT}^{(n+1)T} r(t) \cos(2\pi f_c t + \hat{\phi}) \, dt \int_{nT}^{(n+1)T} r(t) \sin(2\pi f_c t + \hat{\phi}) \, dt = 0$$

This equation suggests the tracking loop configuration illustrated in the following figure.
6.2.5 Non-Decision-Directed Loops

- Non-decision-directed PLL for carrier phase estimation of PAM signals.

Note that the multiplication of the two signals from the integrators destroys the sign carried by the information symbols.

The summer plays the role of the loop filter.
6.2.5 Non-Decision-Directed Loops

**Squaring loop**

- The *squaring loop* is a non-decision-directed loop that is widely used in practice to establish the carrier phase of double-sideband suppressed carrier signals such as PAM.
- Consider the problem of estimating the carrier phase of the digitally modulated PAM signal of the form:

\[
s(t) = A(t) \cos(2\pi f_c t + \phi)
\]

- Note that \( E[s(t)] = E[A(t)] = 0 \) when the signal levels are symmetric about zero.
- One method for generating a carrier from the received signal is to square the signal and, thus, to generate a frequency component at \( 2f_c \), which can be used to drive a PLL tuned to \( 2f_c \).
6.2.5 Non-Decision-Directed Loops

- Squaring loop (cont.)
- Carrier recover using a square-law device
6.2.5 Non-Decision-Directed Loops

Squaring loop (cont.)

- The output of the square-law device is:

\[ s^2(t) = A^2(t) \cos^2(2\pi f_c t + \phi) = \frac{1}{2} A^2(t) + \frac{1}{2} A^2(t) \cos(4\pi f_c t + 2\phi) \]

- Since the modulation is a cyclostationary stochastic process, the expected value of \( s^2(t) \) is:

\[
E \left[ s^2(t) \right] = \frac{1}{2} E \left[ A^2(t) \right] + \frac{1}{2} E \left[ A^2(t) \right] \cos(4\pi f_c t + 2\phi)
\]

- Hence, there is power at the frequency \( 2f_c \).

- The squaring of \( s(t) \) has removed the sign information contained in \( A(t) \) and has resulted in phase-coherent frequency components at twice the carrier.

- The filtered frequency at \( 2f_c \) is used to drive the PLL.
6.2.5 Non-Decision-Directed Loops

Costas loop

Block diagram of Costas loop

\[
\begin{align*}
\times & \quad \cos (2\pi f_c t + \hat{\phi}) \\
90^\circ & \quad \text{phase shift} \\
\times & \quad \sin (2\pi f_c t + \hat{\phi}) \\
\times & \quad e(t) \\
\text{VCO} & \\
\text{Loop filter} & \\
\text{Low-pass filter} & \\
\end{align*}
\]
6.2.5 Non-Decision-Directed Loops

Costas loop (cont.)

The received signal is multiplied by \( \cos(2 \pi f_c t + \hat{\phi}) \) and \( \sin(2 \pi f_c t + \hat{\phi}) \), which are outputs from the VCO. The two products are:

\[
y_c(t) = \left[ s(t) + n(t) \right] \cos\left(2\pi f_c t + \hat{\phi}\right) = \frac{1}{2} \left[ A(t) + n_c(t) \right] \cos \Delta \phi + \frac{1}{2} n_s(t) \sin \Delta \phi + \text{double-frequency terms}
\]

\[
y_s(t) = \left[ s(t) + n(t) \right] \sin\left(2\pi f_c t + \hat{\phi}\right) = \frac{1}{2} \left[ A(t) + n_c(t) \right] \sin \Delta \phi - \frac{1}{2} n_s(t) \cos \Delta \phi + \text{double-frequency terms}
\]
Costas loop (cont.):

- The double-frequency terms are eliminated by the low-pass filters.
- An error signal is generated by multiplying the two outputs of the low-pass filters:

\[
e(t) = \frac{1}{8} \left\{ \left[ A(t) + n_c(t) \right]^2 - n_s^2(t) \right\} \sin(2\Delta \phi)
- \frac{1}{4} n_s(t) \left[ A(t) + n_c(t) \right] \cos(2\Delta \phi)
\]

- This error signal is filtered by the loop filter, whose output is the control voltage that drives the VCO.
- If the loop filter in the Costas loop is identical to that used in the squaring loop, the two loops are equivalent.
6.3 Symbol Timing Estimation

In a digital communication system, the output of the demodulator must be sampled periodically at the symbol rate, at the precise sampling time instants $t_m = mT + \tau$, where

- $T$: symbol interval
- $\tau$: time delay, which accounts for the propagation time of the signal from the transmitter to the receiver.

To perform this periodic sampling, we require a clock signal at the receiver.
6.3 Symbol Timing Estimation

The process of extracting such a clock signal at the receiver is usually called symbol synchronization or timing recovery.
6.3 Symbol Timing Estimation

- Timing recovery:
  - Timing recovery is one of the most critical functions that is performed at the receiver of a synchronous digital communication system.
  - The receiver must know not only the frequency \(1/T\) at which the outputs of the matched filters or correlators are sampled, but also where to take the samples within each symbol interval.
  - The choice of sampling instant within the symbol interval of duration \(T\) is called the *timing phase*. 
6.3 Symbol Timing Estimation

Symbol synchronization:

- In some communication systems, the transmitter and receiver clocks are synchronized to a master clock, which provides a very precise timing signal.
- In this case, the receiver must estimate and compensate for the relative time delay between the transmitted and received signals.
- Such may be the case for radio communication systems that operate in the very low frequency (VLF) band (below 30 kHz), where precise clock signals are transmitted from a master radio station.
6.3 Symbol Timing Estimation

- For the transmitter, we simultaneously transmit the clock frequency $1/T$ or a multiple of $1/T$ along with the information signal.
- The receiver may simply employ a narrowband filter tuned to the transmitted clock frequency, thus, extract the clock signal for sampling.
- Advantage: simple to implement.
- Disadvantages:
  - The transmitter must allocate some of its available power to the transmission of the clock signal.
  - Some small fraction of the available channel bandwidth must be allocated for the transmission of the clock signal.
6.3 Symbol Timing Estimation

- In spite of these disadvantages, this method is frequently used in telephone transmission systems that employ large bandwidths to transmit the signals of many users.
- In such a case, the transmission of a clock signal is shared in the demodulation of the signals among the many users.
- Through this shared use of the clock signal, the penalty in the transmitter power and in bandwidth allocation is reduced proportionally by the number of users.
6.3.1 Maximum-Likelihood Timing Estimation

If the signal is a baseband PAM waveform:

$$r(t) = s(t; \tau) + n(t)$$

where

$$s(t; \tau) = \sum_{n} I_n g(t - nT - \tau)$$

As in the case of ML phase estimation, we distinguish between two types of timing estimators: decision-directed timing estimators and non-decision-directed estimators.
6.3.1 Maximum-Likelihood Timing Estimation

- Decision-directed timing estimators:
  - The information symbols from the output of the demodulator are treated as the known transmitted sequence.
  - In this case, the log-likelihood function has the form:

\[
\Lambda_L(\tau) = C_L \int_{T_0}^{T} r(t)s(t; \tau) \, dt
\]

- Thus, we obtain,

\[
\Lambda_L(\tau) = C_L \sum_n I_n \int_{T_0}^{T} r(t)g(t - nT - \tau) \, dt
\]

\[
= C_L \sum_n I_n y_n(\tau)
\]
6.3.1 Maximum-Likelihood Timing Estimation

where

\[ y_n(\tau) = \int_{T_0} r(t)g(t - nT - \tau) \, dt \]

\( \wedge \)

- A necessary condition for \( \tau \) to be the ML estimate of \( \tau \):

\[
\frac{d \Lambda_L(\tau)}{d\tau} = \sum_n I_n \frac{d}{d\tau} \int_{T_0} r(t)g(t - nT - \tau) \, dt \\
= \sum_n I_n \frac{d}{d\tau} [y_n(\tau)] = 0
\]
6.3.1 Maximum-Likelihood Timing Estimation

- Decision-directed ML estimation of timing for baseband PAM
6.3.1 Maximum-Likelihood Timing Estimation

- We should observe that the summation in the loop serves as the loop filter whose bandwidth is controlled by the length of the sliding window in the summation.
- The output of the loop filter drives the voltage-controlled clock (VCC), or voltage-controlled oscillator, which controls the sampling times for the input of the loop.
- Since the detected information sequence \( \{I_n\} \) is used in the estimation of \( \tau \), the estimate is decision-directed.
A non-decision-directed timing estimate can be obtained by averaging the likelihood ratio $\Lambda(\tau)$ over the PDF of the information symbols, to obtain $\overline{\Lambda}(\tau)$, and then differentiating either $\overline{\Lambda}(\tau)$ or $\ln \overline{\Lambda}(\tau) = \overline{\Lambda}_L(\tau)$ to obtain the condition for the maximum-likelihood estimate $\tau_{ML}$. 
In the case of binary (baseband) PAM, where $I_n = \pm 1$ with equal probability, the average over the data is

$$\bar{\Lambda}_L (\tau) = \sum_n \ln \cosh C_{y_n}(\tau)$$

Since $\ln \cosh x \approx \frac{1}{2} x^2$ for small $x$, the square-law approximation

$$\bar{\Lambda}_L (\tau) \approx \frac{1}{2} C^2 \sum_n y_n^2 (\tau)$$

is appropriate for low signal-to-noise ratios.
For multilevel PAM, we may approximate the statistical characteristics of the information symbols $\{I_n\}$ by the Gaussian PDF, with zero-mean and unit variance.
An implementation of a tracking loop based on the derivative of $\Lambda(\tau)$ is shown as following.
6.3.2 Non-Decision-Directed Timing Estimation

Alternatively, an implementation of a tracking loop based on is shown

\[
\frac{d}{d\tau} \sum_n y_n^2(\tau) = 2 \sum_n y_n(\tau) \frac{dy_n(\tau)}{d\tau} = 0
\]

In both structures, we observe that the summation serves as the loop filter that drives the VCC.
6.3.2 Non-Decision-Directed Timing Estimation

**Early-late gate synchronizers:**

Consider the rectangular pulse $s(t)$, $0 \leq t \leq T$, and the output of the filter matched to $s(t)$ attains its maximum value at time $t = T$:

(a) $s(t)$

(b) Matched filter output

- Early sample
- Optimum sample
- Late sample

0 $T$ $T - \delta$ $T + \delta$ $2T$
Thus, the output of the matched filter is the time autocorrelation function of the pulse $s(t)$.

Of course, it can be applied to any signal pulse.

Clearly, the proper time to sample the output of the matched filter for a maximum output is at $t = T$, i.e., at the peak of the correlation function.

In the presence of noise, the identification of the peak value of the signal is generally difficult.

Instead of sampling the signal at the peak, suppose we sample early at $t = T - \delta$ and late at $t = T + \delta$. The absolute value of the early samples $|y[m(T - \delta)]|$ and the late samples $|y[m(T + \delta)]|$ will be smaller than $|y(mT)|$. 

6.3.2 Non-Decision-Directed Timing Estimation
Since the autocorrelation function is even with respect to the optimum sampling time $t = T$, then

$$|y[m(T - \delta)]| = |y[m(T + \delta)]|$$

Under this condition, the proper sampling time is the midpoint between $t = T - \delta$ and $t = T + \delta$.

This condition forms the basis for the early-late gate symbol synchronizer.
6.3.2 Non-Decision-Directed Timing Estimation

Block diagram of early-late gate synchronizer:
6.3.2 Non-Decision-Directed Timing Estimation

- Correlators are used in place of the equivalent matched filters.
- The two correlators integrate over the symbol interval $T$, but one correlator starts integrating $\delta$ seconds early relative to the estimated optimum sampling time and the other integrator starts integrating $\delta$ seconds late relative to the estimated optimum sampling time.
- An error signal is formed by taking the difference between the absolute values of the two correlator outputs.
To smooth the noise corrupting the signal samples, the error signal is passed through a low-pass filter.

If the timing is off relative to the optimum sampling time, the average error signal at the output of the low-pass filter is nonzero, and the clock signal is either retarded or advanced, depending on the sign of the error.

Thus, the smoothed error signal is used to drive a VCC, whose output is the desired clock signal that is used for sampling.

The output of the VCC is also used as a clock signal for a symbol waveform generator that puts out the same basic pulse waveform as that of the transmitting filter.
6.3.2 Non-Decision-Directed Timing Estimation

- This pulse waveform is advanced and delayed and then fed to the two correlators.

- If the signal pulses are rectangular, there is no need for a signal pulse generator within the tracking loop.

- We observe that the early-late gate synchronizer is basically a closed-loop control system whose bandwidth is relatively narrow compared to the symbol rate $1/T$.

- The bandwidth of the loop determines the quality of the timing estimate.
A narrowband loop provides more averaging over the additive noise, and thus, improves the quality of the estimated sampling instants, provided that the channel propagation delay is constant and the clock oscillator at the transmitter is not drifting with time.

On the other hand, if the channel propagation delay is changing with time and/or the transmitter clock is also drifting with time, then the bandwidth of the loop must be increased to provide for faster tracking of time variations in symbol timing.
In the tracking mode, the two correlators are affected by adjacent symbols. However, if the sequence of information symbols has zero-mean, as is the case for PAM and some other signal modulations, the contribution to the output of the correlators from adjacent symbols averages out to zero in the low-pass filter.
6.3.2 Non-Decision-Directed Timing Estimation

An equivalent realization of the early-late gate synchronizer:
6.3.2 Non-Decision-Directed Timing Estimation

- The clock signal from the VCC is advanced and delayed by $\delta$, and these clock signals are used to sample the outputs of the two correlators.

- The early-late gate synchronizer is a non-decision-directed estimator of symbol timing that approximates the ML estimator.

**proof:**

- By approximating the derivative of the log-likelihood function by the finite difference, i.e.,

$$\frac{d \Lambda_L(\tau)}{d\tau} \approx \frac{\Lambda_L(\tau + \delta) - \Lambda_L(\tau - \delta)}{2\delta}$$
Thus, we obtain

$$\Lambda_L(\tau) \approx \frac{1}{2} C^2 \sum_n y_n^2(\tau)$$

$$\frac{d \Lambda_L(\tau)}{d\tau} = \frac{C^2}{4\delta} \sum_n \left[ y_n^2(\tau + \delta) - y_n^2(\tau - \delta) \right]$$

$$\approx \frac{C^2}{4\delta} \sum_n \left\{ \left[ \int_{t_0} r(t) g(t - nT - \tau - \delta) \, dt \right]^2 \right\}$$