Chapter 4
Characterization of Communication Signals and Systems

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4.1 Representation of Band-Pass Signals and Systems

- The channel over which the signal is transmitted is limited in bandwidth to an interval of frequencies centered about the carrier.

- Signals and channels (systems) that satisfy the condition that their bandwidth is much smaller than the carrier frequency are termed narrowband band-pass signals and channels (systems).

- With no loss of generality and for mathematical convenience, it is desirable to reduce all band-pass signals and channels to equivalent low-pass signals and channels.
4.1.1 Representation of Band-Pass Signals

Suppose that a real-valued signal \( s(t) \) has a frequency content concentrated in a narrow band of frequencies in the vicinity of a frequency \( f_c \), as shown in the following figure:

Our object is to develop a mathematical representation of such signals.
4.1.1 Representation of Band-Pass Signals

A signal that contains only the positive frequencies in $s(t)$ may be expressed as:

$$S_+(f) = 2u(f)S(f)$$

$$s_+(t) = \int_{-\infty}^{\infty} S_+(f) \cdot e^{j2\pi ft} df$$

$$= F^{-1} \left[ 2u(f) \right] * F^{-1} \left[ S(f) \right]$$

where $S(f)$ is the Fourier transform of $s(t)$ and $u(f)$ is the unit step function, and the signal $s_+(t)$ is called the analytic signal or the pre-envelope of $s(t)$.

$$F^{-1} \left[ 2u(f) \right] = \delta(t) + \frac{j}{\pi t}$$

$$s_+(t) = \left[ \delta(t) + \frac{j}{\pi t} \right] * s(t) = s(t) + \frac{1}{\pi t} * s(t) \equiv s(t) + j\hat{s}(t)$$
Define:

\[ \hat{s}(t) = \frac{1}{\pi t} * s(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{t - \tau} \, d\tau \]

A filter, called a \textit{Hilbert transformer}, is defined as:

\[ h(t) = \frac{1}{\pi t}, \quad -\infty < t < \infty \]

The signal \( \hat{s}(t) \) may be viewed as the output of the Hilbert transformer when excited by the input signal \( s(t) \).

The frequency response of this filter is:

\[ H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} \, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t} e^{-j2\pi ft} \, dt = \begin{cases} -j & (f > 0) \\ 0 & (f = 0) \\ j & (f < 0) \end{cases} \]
4.1.1 Representation of Band-Pass Signals

- We observe that $|H(f)|=1$ and the phase response $\Theta(f)=-\pi/2$ for $f>0$ and $\Theta(f)=\pi/2$ for $f<0$. Thus, this filter is basically a 90 degrees phase shifter for all frequencies in the input signal.

- The analytic signal $s_+(t)$ is a band-pass signal. To obtain an equivalent low-pass representation, we define:

$$S_l(f) = S_+(f + f_c)$$

$$s_l(t) = s_+(t)e^{-j2\pi f_ct} = \left[s(t) + j\hat{s}(t)\right]e^{-j2\pi f_ct}$$

- In general, $s_l(t)$ is complex-valued:

$$s_l(t) = x(t) + j y(t)$$

$$s(t) = x(t)\cos 2\pi f_ct - y(t)\sin 2\pi f_ct$$

$$\hat{s}(t) = x(t)\sin 2\pi f_ct + y(t)\cos 2\pi f_ct$$
4.1.1 Representation of Band-Pass Signals

- \( s(t) = x(t)\cos2\pi f_c t - y(t)\sin2\pi f_c t \) is the desired form for the representation of a band-pass signal. The low-frequency signal components \( x(t) \) and \( y(t) \) may be viewed as amplitude modulations impressed on the carrier components \( \cos2\pi f_c t \) and \( \sin2\pi f_c t \), respectively.
- \( x(t) \) and \( y(t) \) are called the *quadrature components* of the band-pass signal \( s(t) \).
- \( s(t) \) can also be written as:
  \[
  s(t) = \text{Re} \left\{ x(t) + j y(t) e^{j2\pi f_c t} \right\} = \text{Re} \left\{ s_l(t) e^{j2\pi f_c t} \right\}
  \]
- The low pass signal \( s_l(t) \) is usually called the *complex envelope* of the real signal \( s(t) \) and is basically the *equivalent low-pass signal*. 

\[ \text{Re} \left\{ s_l(t) e^{j2\pi f_c t} \right\} \]
4.1.1 Representation of Band-Pass Signals

- $s_I(t)$ can be also be written as:

$$s_I(t) = a(t) e^{j\theta(t)}$$

where $a(t) = \sqrt{x^2(t) + y^2(t)}$ and $\theta(t) = \tan^{-1} \frac{y(t)}{x(t)}$

- $s(t)$ can be represented as:

$$s(t) = \text{Re}\left[s_I(t) e^{j2\pi f_c t}\right] = \text{Re}\left[a(t) e^{j[2\pi f_c t + \theta(t)]}\right]$$

$$= a(t) \cos\left[2\pi f_c t + \theta(t)\right]$$

$a(t)$ is called the envelope of $s(t)$, and $\theta(t)$ is called the phase of $s(t)$. 
4.1.1 Representation of Band-Pass Signals

Three equivalent representations of band-pass signals:

\[ s(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \]
\[ = \text{Re} \left[ s_l(t) e^{j2\pi f_c t} \right] \]
\[ = a(t) \cos \left[ 2\pi f_c t + \theta(t) \right] \]

The Fourier transform of \( s(t) \) is:

\[ S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \left\{ \text{Re} \left[ s_l(t) e^{j2\pi f_c t} \right] \right\} e^{-j2\pi ft} dt \]
\[ S(f) = \frac{1}{2} \int_{-\infty}^{\infty} \left[ s_l(t) e^{j2\pi f_c t} + s_l^*(t) e^{-j2\pi f_c t} \right] e^{-j2\pi ft} dt \]
\[ = \frac{1}{2} \left[ S_l(f - f_c) + S_l^*(-f - f_c) \right] \]

\[ \text{Re}(\xi) = \frac{1}{2} \left( \xi + \xi^* \right) \]
The energy in the signal \( s(t) \) is defined as:

\[
\mathcal{E} = \int_{-\infty}^{\infty} s^2(t) \, dt = \int_{-\infty}^{\infty} \left\{ \text{Re} \left[ s_l(t) e^{j2\pi f_c t} \right] \right\}^2 \, dt
\]

\[
\mathcal{E} = \frac{1}{4} \int_{-\infty}^{\infty} \left[ s_l^2 e^{j4\pi f_c t} + 2s_ls_l^* + \left( s_l^* \right)^2 e^{-j4\pi f_c t} \right] \, dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \left| s_l(t) \right|^2 \, dt + \frac{1}{4} \int_{-\infty}^{\infty} \left[ a^2(t) e^{j4\pi f_c t + 2\theta(t)} + (a^*(t))^2 e^{-(j4\pi f_c t + 2\theta(t))} \right] \, dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \left| s_l(t) \right|^2 \, dt + \frac{1}{2} \int_{-\infty}^{\infty} \left| s_l(t) \right|^2 \cos \left( 4\pi f_c t + 2\theta(t) \right) \, dt
\]

where \( \left| s_l(t) \right|^2 = a^2(t) = (a^*(t))^2 \)
Since the signal $s(t)$ is narrow-band, the real envelope $a(t) = |s_l(t)|$
or, equivalently, $a^2(t)$ varies slowly relative to the rapid
variations exhibited by the cosine function.

The net area contributed by the second integral is very small
relative to the value of the first integral, hence, it can be
neglected.

$$\varepsilon = \frac{1}{2} \int_{-\infty}^{\infty} \left| s_l(t) \right|^2 dt \quad (4.1-24)$$
A linear filter or system may be described either by its impulse response $h(t)$ or by its frequency response $H(f)$, which is the Fourier transform of $h(t)$. Since $h(t)$ is real, $H^*(-f) = H(f)$, because:

$$H^*(-f) = \left( \int_{-\infty}^{\infty} h(t) e^{-j2\pi(-f)t} \, dt \right)^*$$

$$= \int_{-\infty}^{\infty} h^*(t) e^{-j2\pi ft} \, dt$$

Note that $h(t)$ is real.

$$= \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} \, dt = H(f)$$

Define $H_l(f - f_c) = \begin{cases} H(f) & (f > 0) \\ 0 & (f < 0) \end{cases}$

then $H_l^*(-f - f_c) = \begin{cases} 0 & (f > 0) \\ H^*(-f) = H(f) & (f < 0) \end{cases}$
4.1.2 Representation of Linear Band-Pass Systems

As a result

\[ H(f) = H_l(f - f_c) + H_l^*(-f - f_c) \]

thus

\[ h(t) = h_l(t) e^{j2\pi f_c t} + h_l^* e^{-j2\pi f_c t} \]

\[ = 2 \text{Re} \left[ h_l(t) e^{j2\pi f_c t} \right] \]

where

\[ \int_{-\infty}^{\infty} H_l^*(-f - f_c) e^{j2\pi ft} df \]

using \( x = -f - f_c \)

\[ = \int_{-\infty}^{\infty} H_l^*(x) e^{-j2\pi xt} dx \cdot e^{-j2\pi f_c t} \]

\[ = \left( \int_{-\infty}^{\infty} H_l(x) e^{j2\pi xt} dx \right)^* \cdot e^{-j2\pi f_c t} = h_l^*(t) \cdot e^{-j2\pi f_c t} \]

In general, the impulse response \( h_l(t) \) of the equivalent low-pass system is complex-valued.
4.1.3 Response of a Band-Pass System to a Band-Pass Signal

- We have shown in Sections 4.1.1 and 4.1.2 that narrow band band-pass signals and systems can be represented by equivalent low-pass signals and systems.

- We demonstrate in this section that the output of a band-pass system to a band-pass input signal is simply obtained from the equivalent low-pass input signal and the equivalent low-pass impulse response of the system.

- The output of the band-pass system is also a band-pass signal, and, therefore, it can be expressed in the form:
  \[ r(t) = \int_{-\infty}^{\infty} s(\tau) h(t-\tau) d\tau \]
  where \( r(t) \) is related to the input signal \( s(t) \) and the impulse response \( h(t) \) by the convolution integral.
  \[ r(t) = \text{Re} \left[ r_l(t) e^{j2\pi f_c t} \right] \]
4.1.3 Response of a Band-Pass System to a Band-Pass Signal

The output of the system in the frequency domain is:

$$R(f) = S(f)H(f)$$


$$= \frac{1}{2} \left[ S_l(f-f_c) + S_l^*(-f-f_c) \right] \left[ H_l(f-f_c) + H_l^*(-f-f_c) \right]$$

- For a narrow band signal, $S_l(f-f_c) \approx 0$ for $f < 0$ and $H_l^*(-f-f_c) = 0$ for $f > 0$.

$$S_l(f-f_c)H_l^*(-f-f_c) = 0$$

4.1-27

- For a narrow band signal, $S_l^*(-f-f_c) \approx 0$ for $f > 0$ and $H_l(f-f_c) = 0$ for $f < 0$.

$$S_l^*(-f-f_c)H_l(f-f_c) = 0$$

4.1-26

$$R(f) = \frac{1}{2} \left[ S_l(f-f_c)H_l(f-f_c) + S_l^*(-f-f_c)H_l^*(-f-f_c) \right]$$

$$= \frac{1}{2} \left[ R_l(f-f_c) + R_l^*(-f-f_c) \right]$$

$$R_l(f) \equiv S_l(f)H_l(f)$$

$$r_l(t) = \int_{-\infty}^{\infty} s_l(\tau)h_l(t-\tau) \, d\tau$$
In this section, we extend the representation to sample functions of a *band-pass stationary stochastic process*. In particular, we derive the relations between the correlation functions and power spectra of the band-pass signal and the correlation function and power spectra of the equivalent low-pass signal.

Suppose that $n(t)$ is a sample function of a wide-sense stationary stochastic process with zero mean and power spectral density $\Phi_{nn}(f)$. The power spectral density is assumed to be zero outside of an interval of frequencies centered around $f_c$, where $f_c$ is termed the *carrier frequency*. The stochastic process $n(t)$ is said to be a *narrowband band-pass process* if the width of the spectral density is much smaller than $f_c$. 

4.1.4 Representation of Band-Pass Stationary Stochastic Processes
4.1.4 Representation of Band-Pass Stationary Stochastic Processes

Under this condition, a sample function of the process $n(t)$ can be represented by the following equations from Section 4.1.1:

$$n(t) = a(t)\cos\left(2\pi f_c t + \theta(t)\right)$$

$$= x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t$$

$$= \text{Re}\left[z(t)e^{j2\pi f_c t}\right]$$

- $a(t)$ is the envelope and $\theta(t)$ is the phase of the real-valued signal.
- $x(t)$ and $y(t)$ are the quadrature components of $n(t)$.
- $z(t)$ is called the complex envelope of $n(t)$.
- If $n(t)$ is zero mean, then $x(t)$ and $y(t)$ must also have zero mean values.
- The stationarity of $n(t)$ implies that:

$$\phi_{xx}(\tau) = \phi_{yy}(\tau)$$

$$\phi_{xy}(\tau) = -\phi_{yx}(\tau)$$

Proved next.
4.1.4 Representation of Band-Pass Stationary Stochastic Processes

Proof of \( \phi_{xx}(\tau) = \phi_{yy}(\tau) \) and \( \phi_{xy}(\tau) = -\phi_{yx}(\tau) \)

Autocorrelation function of \( n(t) \) is:
\[
\phi_{nn}(\tau) = E \left[ n(t) n(t + \tau) \right] = E \left\{ \left[ x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \right] \times \left[ x(t + \tau) \cos 2\pi f_c (t + \tau) - y(t + \tau) \sin 2\pi f_c (t + \tau) \right] \right\}
\]
\[
= \phi_{xx}(\tau) \cos 2\pi f_c t \cos 2\pi f_c (t + \tau) + \phi_{yy}(\tau) \sin 2\pi f_c t \sin 2\pi f_c (t + \tau)
- \phi_{xy}(\tau) \sin 2\pi f_c t \cos 2\pi f_c (t + \tau) - \phi_{yx}(\tau) \cos 2\pi f_c t \sin 2\pi f_c (t + \tau)
\]

by using:
\[
\cos A \cos B = \frac{1}{2} \left[ \cos (A - B) + \cos (A + B) \right]
\]
\[
\sin A \sin B = \frac{1}{2} \left[ \cos (A - B) - \cos (A + B) \right]
\]
\[
\sin A \cos B = \frac{1}{2} \left[ \sin (A - B) + \sin (A + B) \right]
\]
4.1.4 Representation of Band-Pass Stationary Stochastic Processes

We can obtain:

\[ \phi_{nn}(\tau) = E\left[ n(t)n(t+\tau) \right] \]

\[ = \frac{1}{2} \left[ \phi_{xx}(\tau) + \phi_{yy}(\tau) \right] \cos 2\pi f_c \tau + \frac{1}{2} \left[ \phi_{xx}(\tau) - \phi_{yy}(\tau) \right] \cos 2\pi f_c (2t+\tau) \]

\[ - \frac{1}{2} \left[ \phi_{yx}(\tau) - \phi_{xy}(\tau) \right] \sin 2\pi f_c \tau - \frac{1}{2} \left[ \phi_{yx}(\tau) + \phi_{xy}(\tau) \right] \sin 2\pi f_c (2t+\tau) \]

Since \( n(t) \) is stationary, the right-hand side must be independent of \( t \).

As a result, \( \phi_{xx}(\tau) = \phi_{yy}(\tau) \) and \( \phi_{xy}(\tau) = -\phi_{yx}(\tau) \) \[ \text{Q.E.D.} \]

Therefore, \[ \phi_{nn}(\tau) = \phi_{xx}(\tau) \cos 2\pi f_c \tau - \phi_{yx}(\tau) \sin 2\pi f_c \tau \]

Note that this equation is identical in form to:

\[ n(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \]
4.1.4 Representation of Band-Pass Stationary Stochastic Processes

The autocorrelation function of the equivalent low-pass process $z(t)=x(t)+jy(t)$ is defined as:

$$
\phi_{zz}(\tau) = \frac{1}{2} E \left[ z^*(t) z(t+\tau) \right]
$$

$$
\phi_{zz}(\tau) = \frac{1}{2} \left[ \phi_{xx}(\tau) + \phi_{yy}(\tau) - j\phi_{xy}(\tau) + j\phi_{yx}(\tau) \right]
$$

Since $\phi_{xx}(\tau) = \phi_{yy}(\tau)$ and $\phi_{xy}(\tau) = -\phi_{yx}(\tau)$
we obtain: $\phi_{zz}(\tau) = \phi_{xx}(\tau) + j\phi_{yx}(\tau)$

This equation relates the autocorrelation function of the complex envelope to the autocorrelation and cross-correlation functions of the quadrature components.
By combining $\phi_{zz}(\tau) = \phi_{xx}(\tau) + j\phi_{yx}(\tau)$
and
$\phi_{nn}(\tau) = \phi_{xx}(\tau) \cos 2\pi f_c \tau - \phi_{yx}(\tau) \sin 2\pi f_c \tau$
we can obtain:

$$\phi_{nn}(\tau) = \text{Re}[\phi_{zz}(\tau) e^{j2\pi f_c \tau}]$$

Therefore, the autocorrelation function $\phi_{nn}(\tau)$ of the band-pass stochastic process is uniquely determined from the autocorrelation function $\phi_{zz}(\tau)$ of the equivalent low-pass process $z(t)$ and the carrier frequency $f_c$.

The power density spectrum of the stochastic process $n(t)$ is:

$$\Phi_{nn}(f) = \int_{-\infty}^{\infty} \left\{ \text{Re} \left[ \phi_{zz}(\tau) e^{j2\pi f_c \tau} \right] \right\} e^{-j2\pi f \tau} d\tau$$

$$= \frac{1}{2} \left[ \Phi_{zz}(f - f_c) + \Phi_{zz}(-f - f_c) \right]$$
4.1.4 Representation of Band-Pass Stationary Stochastic Processes

Properties of the quadrature components

- \( \phi_{yx}(\tau) = \phi_{xy}(-\tau) \quad (2.2-10) \)
- \( \phi_{xy}(\tau) = -\phi_{yx}(\tau) \quad (4.1-41) \)

\[ \Rightarrow \quad \phi_{xy}(\tau) = -\phi_{xy}(-\tau) \]

\[ \phi_{zz}(\tau) = \frac{1}{2} \left[ \phi_{xx}(\tau) + \phi_{yy}(\tau) - j\phi_{xy}(\tau) + j\phi_{yx}(\tau) \right] \]

\[ \Rightarrow \quad \phi_{xy}(\tau) \text{ is an odd function of } \tau \text{ and } \phi_{xy}(0) = 0. \]

- If \( \phi_{xy}(\tau) = 0 \text{ for all } \tau, \) then \( \phi_{zz}(\tau) \) is real (from 4.1-48) and the power spectral density satisfies \( \Phi_{zz}(f) = \Phi_{zz}(-f) \)
  (i.e. \( \Phi_{zz}(f) \) is symmetric about \( f = 0 \)).

- If \( n(t) \) is Gaussian, \( x(t) \) and \( y(t + \tau) \) are jointly Gaussian. For \( \tau = 0 \), they are statistically independent, and the joint PDF is:

\[ p(x, y) = \frac{1}{2\pi\sigma^2} e^{-\left(x^2 + y^2\right)/2\sigma^2}, \quad \sigma^2 = \phi_{xx}(0) = \phi_{yy}(0) = \phi_{nn}(0) \]
Representation of white noise

White noise is a stochastic process that is defined to have a flat (constant) power spectral density over the entire frequency range. This type of noise can’t be expressed in terms of quadrature components, as a result of its wideband character.

In the demodulation of narrowband signals in noise, it is mathematically convenient to model the additive noise process as white and to represent the noise in terms of quadrature components. This can be accomplished by postulating that the signals and noise at the receiving terminal have passed through an ideal band-pass filter.
4.1.4 Representation of Band-Pass Stationary Stochastic Processes

- Representation of white noise (cont.)

  - The noise resulting from passing the white noise process through a spectrally band-pass filter is termed **band-pass white noise** and has the power spectral density:

    \[ \Phi_{nn}(f) \]

  - The band-pass white noise can be represented by:

    \[
    n(t) = a(t) \cos\left(2\pi f_c t + \theta(t)\right) \\
    = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \\
    = \text{Re}\left[z(t) e^{j2\pi f_c t}\right]
    \]
4.1.4 Representation of Band-Pass Stationary Stochastic Processes

Representation of white noise (cont.)

- The equivalent low-pass noise \( z(t) \) has a power spectral density:
  \[
  \Phi_{zz}(f) = \begin{cases} 
  N_0 \left| f \right| \leq \frac{1}{2} B \\ 
  0 & \left| f \right| > \frac{1}{2} B
  \end{cases}
  \]

\[
\phi_{zz}(\tau) = N_0 \frac{\sin \pi B \tau}{\pi \tau} \quad \text{and} \quad \phi_{zz}(\tau) = N_0 \delta(\tau)
\]

- The power spectral density for white noise and band-pass white noise is symmetric about \( f = 0 \), so \( \phi_{yx}(\tau) = 0 \) for all \( \tau \).

\[
\phi_{zz}(\tau) = \phi_{xx}(\tau) = \phi_{yy}(\tau) \quad \text{(from 4.1-48)}
\]
4.2 Signal Space Representations

We will demonstrate that signals have characteristics that are similar to vectors and develop a vector representation for signal waveforms.

4.2.1 Vector Space Concepts

- A vector \( \mathbf{v} \) in an \( n \)-dimensional space is characterized by its \( n \) components \([v_1 \ v_2 \ \cdots \ v_n]\) and may also be represented as a linear combination of unit vectors or basis vectors \( \mathbf{e}_i \), \( 1 \leq i \leq n \),

\[
\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i
\]

- The *inner product* of two \( n \)-dimensional vectors is defined as:

\[
\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum_{i=1}^{n} v_{1i} v_{2i}
\]
4.2.1 Vector Space Concepts

- A set of $m$ vectors $v_k$, $1 \leq k \leq m$ are orthogonal if:
  \[ v_i \cdot v_j = 0 \text{ for all } 1 \leq i, j \leq m, \text{ and } i \neq j. \]

- The *norm* of a vector $v$ is denoted by $\|v\|$ and is defined as:
  \[ \|v\| = (v \cdot v)^{1/2} = \sqrt{\sum_{i=1}^{n} v_i^2} \]

- A set of $m$ vectors is said to be *orthonormal* if the vectors are orthogonal and each vector has a unit norm.

- A set of $m$ vectors is said to be *linearly independent* if no one vector can be represented as a linear combination of the remaining vectors.

- Two $n$-dimensional vectors $v_1$ and $v_2$ satisfy the triangle inequality:
  \[ \|v_1 + v_2\| \leq \|v_1\| + \|v_2\| \]
**4.2.1 Vector Space Concepts**

- **Cauchy-Schwarz inequality:**
  \[ |\mathbf{v}_1 \cdot \mathbf{v}_2| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| \]

- The norm square of the sum of two vectors may be expressed as:
  \[ \|\mathbf{v}_1 + \mathbf{v}_2\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + 2\mathbf{v}_1 \cdot \mathbf{v}_2 \]

- **Linear transformation** in an \( n \)-dimensional vector space:
  \[ \mathbf{v}' = A\mathbf{v} \]

- In the special case where \( \mathbf{v}' = \lambda \mathbf{v} \), \( A\mathbf{v} = \lambda \mathbf{v} \)
  the vector \( \mathbf{v} \) is called an *eigenvector* and \( \lambda \) is the corresponding *eigenvalue*. 
**Gram-Schmidt procedure** for constructing a set of orthonormal vectors.

- Arbitrarily selecting a vector $v_1$ and normalizing its length:
  \[ u_1 = \frac{v_1}{\|v_1\|} \]

- Select $v_2$ and subtract the projection of $v_2$ onto $u_1$.
  \[ u_2' = v_2 - (v_2 \cdot u_1)u_1 \]

- Normalize the vector $u_2'$ to unit length.
  \[ u_2 = \frac{u_2'}{\|u_2'\|} \]

- Selecting $v_3$: $u_3' = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2$
  \[ u_3 = \frac{u_3'}{\|u_3'\|} \]

- By continuing this procedure, we construct a set of orthonormal vectors.
4.2.2 Signal Space Concepts

- The **inner product** of two generally complex-valued signals \(x_1(t)\) and \(x_2(t)\) is denote by \(<x_1(t), x_2(t)>\) and defined as:

\[
\langle x_1(t), x_2(t) \rangle = \int_a^b x_1(t)x_2^*(t)\,dt
\]

- The signals are **orthogonal** if their inner product is zero.

- The **norm** of a signal is defined as:

\[
\|x(t)\| = \left(\int_a^b |x(t)|^2\,dt\right)^{1/2}
\]

- A set of \(m\) signals are **orthonormal** if they are orthogonal and their norms are all unity.

- A set of \(m\) signals is **linearly independent**, if no signal can be represented as a linear combination of the remaining signals.
4.2.2 Signal Space Concepts

- The **triangle inequality** for two signals is:
  \[ \| x_1(t) + x_2(t) \| \leq \| x_1(t) \| + \| x_2(t) \| \]

- The **Cauchy-Schwarz inequality** is:
  \[ \left| \int_a^b x_1(t) x_2^*(t) \, dt \right| \leq \left( \int_a^b |x_1(t)|^2 \, dt \right)^{1/2} \left( \int_a^b |x_2(t)|^2 \, dt \right)^{1/2} \]

  with equality when \( x_2(t) = ax_1(t) \), where \( a \) is any complex number.
Suppose that $s(t)$ is a deterministic, real-valued signal with finite energy:

$$\varepsilon_s = \int_{-\infty}^{\infty} \left[ s(t) \right]^2 dt$$

Suppose that there exists a set of functions $\{f_n(t), n=1,2,\ldots,K\}$ that are orthonormal in the sense that:

$$\int_{-\infty}^{\infty} f_n(t) f_m(t) dt = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases}$$

We may approximate the signal $s(t)$ by a weighted linear combination of these functions, i.e.,

$$s(t) = \sum_{k=1}^{K} s_k f_k(t)$$
The approximation error incurred is:

\[ e(t) = s(t) - \hat{s}(t) \]

The energy of the approximation error:

\[ \mathcal{E}_e = \int_{-\infty}^{\infty} \left[ s(t) - \hat{s}(t) \right]^2 dt = \int_{-\infty}^{\infty} \left[ s(t) - \sum_{k=1}^{K} s_k f_k(t) \right]^2 dt \quad (4.2-24) \]

To minimize the energy of the approximation error, the optimum coefficients in the series expansion of \( s(t) \) may be found by:

- Differentiating Equation 4.2-24 with respect to each of the coefficients \( \{s_k\} \) and setting the first derivatives to zero.
- Use a well-known result from estimation theory based on the mean-square-error criterion, which is that the minimum of \( \mathcal{E}_e \) with respect to the \( \{s_k\} \) is obtained when the error is orthogonal to each of the functions in the series expansion.
Using the second approach, we have:

$$\int_{-\infty}^{\infty} \left[ s(t) - \sum_{k=1}^{K} s_k f_k(t) \right] f_n(t) \, dt = 0, \quad n = 1, 2, \ldots, K$$

Since the functions \( \{f_n(t)\} \) are orthonormal, we have:

$$s_n = \int_{-\infty}^{\infty} s(t) f_n(t) \, dt, \quad n = 1, 2, \ldots, K$$

Thus, the coefficients are obtained by projecting the signals \( s(t) \) onto each of the functions.

The minimum mean square approximation error is:

$$\epsilon_{\text{min}} = \int_{-\infty}^{\infty} e(t) s(t) \, dt = \int_{-\infty}^{\infty} \left[ s(t) \right]^2 \, dt - \int_{-\infty}^{\infty} \sum_{k=1}^{K} s_k f_k(t) s(t) \, dt$$

$$= \epsilon_s - \sum_{k=1}^{K} s_k^2$$
4.2.3 Orthogonal Expansions of Signals

- When the minimum mean square approximation error $\varepsilon_{\text{min}} = 0$,

\[ \varepsilon_s = \sum_{k=1}^{K} s_k^2 = \int_{-\infty}^{\infty} \left[ s(t) \right]^2 dt \]

- Under such condition, we may express $s(t)$ as:

\[ s(t) = \sum_{k=1}^{K} s_k f_k(t) \]

- When every finite energy signal can be represented by a series expansion of the form for which $\varepsilon_{\text{min}} = 0$, the set of orthonormal functions $\{f_n(t)\}$ is said to be complete.
Example 4.2-1 Trigonometric Fourier Series:

Consider a finite energy signal $s(t)$ that is zero everywhere except in the range $0 \leq t \leq T$ and has a finite number of discontinuities in this interval. Its periodic extension can be represented in a Fourier series as:

$$s(t) = \sum_{k=0}^{\infty} \left( a_k \cos \frac{2\pi kt}{T} + b_k \sin \frac{2\pi kt}{T} \right)$$

where the coefficients $\{a_k, b_k\}$ that minimize the mean square error are given by:

$$a_k = \frac{1}{\sqrt{T}} \int_{0}^{T} s(t) \cos \frac{2\pi kt}{T} \, dt, \quad b_k = \frac{1}{\sqrt{T}} \int_{0}^{T} s(t) \sin \frac{2\pi kt}{T} \, dt$$

The set of trigonometric functions $\left\{ \sqrt{2/T} \cos 2\pi kt / T, \sqrt{2/T} \sin 2\pi kt / T \right\}$ is complete, and the series expansion results in zero mean square error.
Gram-Schmidt procedure

- Constructing a set of orthonormal waveforms from a set of finite energy signal waveforms \( \{ s_i(t), \, i=1,2,\cdots,M \} \).
- Begin with the first waveform \( s_1(t) \) which has energy \( \varepsilon_1 \).

The first orthonormal waveform is:

\[
 f_1(t) = \frac{s_1(t)}{\sqrt{\varepsilon_1}}
\]

- The 2nd waveform is constructed from \( s_2(t) \) by first computing the projection of \( f_1(t) \) onto \( s_2(t) \):

\[
 c_{12} = \int_{-\infty}^{\infty} s_2(t) f_1(t) \, dt
\]

- Then \( c_{12} f_1(t) \) is subtracted from \( s_2(t) \):

\[
 f_2(t) = s_2(t) - c_{12} f_1(t)
\]
Gram-Schmidt procedure (cont.)

- If $\varepsilon_2$ denotes the energy of $f_2'(t)$, the normalized waveform that is orthogonal to $f_1(t)$ is:
  \[ f_2(t) = \frac{f_2'(t)}{\sqrt{\varepsilon_2}} \]

- In general, the orthogonalization of the $k$th function leads to
  \[ f_k(t) = \frac{f_k'(t)}{\sqrt{\varepsilon_k}} \]
  where $f_k'(t) = s_k(T) - \sum_{i=1}^{k-1} c_{ik} f_i(t)$

- The orthogonalization process is continued until all the $M$ signal waveforms have been exhausted and $N \leq M$ orthonormal waveforms have been constructed.
Gram-Schmidt procedure (cont.)

Once we have constructed the set of orthonormal waveforms \( \{ f_n(t) \} \), we can express the \( M \) signals \( \{ s_n(t) \} \) as linear combinations of the \( \{ f_n(t) \} \):\

\[
 s_k(t) = \sum_{n=1}^{N} s_{kn} f_n(t), \quad k = 1, 2, \ldots, M
\]

\[
 \varepsilon_k = \int_{-\infty}^{\infty} \left[ s_k(t) \right]^2 dt = \sum_{n=1}^{N} s_{kn}^2 = ||s_k||^2
\]

\[
 s_k = \begin{bmatrix}
 s_{k1} & s_{k2} & \cdots & s_{kN}
 \end{bmatrix}
\]

Each signal may be represented as a point in the \( N \)-dimensional signal space with coordinates \( \{ s_{ki}, i=1,2,\ldots,N \} \).
4.2.3 Orthogonal Expansions of Signals

Gram-Schmidt procedure (cont.)

- The energy in the $k$th signal is simply the square of the length of the vector or, equivalently, the square of the Euclidean distance from the origin to the point in the $N$-dimensional space.
- Any signal can be represented geometrically as a point in the signal space spanned by the $\{f_n(t)\}$.
- The functions $\{f_n(t)\}$ obtained from the Gram-Schmidt procedure are not unique.
- If we alter the order in which the orthogonalization of the signals $\{s_n(t)\}$ is performed, the orthonormal waveforms will be different.
- Nevertheless, the vectors $\{s_n(t)\}$ will retain their geometrical configuration and their lengths will be invariant to the choice of orthonormal functions $\{f_n(t)\}$.
Consider the case in which the signal waveforms are band-pass and represented as:

\[ s_m(t) = \text{Re} \left[ s_{lm}(t) e^{j2\pi f_c t} \right], \quad m = 1, 2, \ldots, M \]

\[ \varepsilon_m = \int_{-\infty}^{\infty} s_m^2(t) \, dt = \frac{1}{2} \int_{-\infty}^{\infty} |s_m(t)|^2 \, dt \quad \text{(from 4.1-24)} \]

Similarity between any pair of signal waveforms is measured by the **normalized cross correlation**:

\[ \frac{1}{\sqrt{\varepsilon_m \varepsilon_k}} \int_{-\infty}^{\infty} s_m(t) s_k(t) \, dt = \text{Re} \left\{ \frac{1}{2 \sqrt{\varepsilon_m \varepsilon_k}} \int_{-\infty}^{\infty} s_{lm}(t) s_{lk}^*(t) \, dt \right\} \quad (4.2-44) \]

**Complex-valued cross-correlation coefficient** \( \rho_{km} \) is defined as:

\[ \rho_{km} = \frac{1}{\sqrt{\varepsilon_m \varepsilon_k}} \int_{-\infty}^{\infty} s_{lm}^*(t) s_{lk}(t) \, dt \quad (4.2-45) \]
The Euclidean distance between a pair of signals is defined as:

$$d_{km}^{(e)} = \| s_m - s_k \| = \left( \int_{-\infty}^{\infty} \left[ s_m(t) - s_k(t) \right]^2 dt \right)^{1/2}$$

$$= \left\{ \varepsilon_m + \varepsilon_k - 2\sqrt{\varepsilon_m \varepsilon_k} \Re(\rho_{km}) \right\}^{1/2}$$

When \( \varepsilon_m = \varepsilon_k = \varepsilon \) for all \( m \) and \( k \), this expression simplifies to:

$$d_{km}^{(e)} = \left\{ 2\varepsilon \left[ 1 - \Re(\rho_{km}) \right] \right\}^{1/2}$$
4.3 Representation of Digitally Modulated Signals

- Digitally modulated signals, which are classified as linear, are conveniently expanded in terms of two orthonormal basis functions of the form:

\[
\begin{align*}
    f_1(t) &= \sqrt{\frac{2}{T}} \cos 2\pi f_c t \\
    f_2(t) &= -\sqrt{\frac{2}{T}} \sin 2\pi f_c t
\end{align*}
\]

- If \( s_{lm}(t) \) is expressed as \( s_{lm}(t) = x_l(t) + jy_l(t) \), \( s_m(t) \) may be expressed as:

\[
    s_m(t) = x_l(t)f_1(t) + y_l(t)f_2(t)
\]

- From 4.2-42.

- In the transmission of digital information over a communication channel, the modulator is the interface device that maps the digital information into analog waveforms that match the characteristics of the channel.
The mapping is generally performed by taking blocks of $k=\log_2 M$ binary digits at a time from the information sequence $\{a_n\}$ and selecting one of $M=2^k$ deterministic, finite energy waveforms $\{s_m(t), m=1,2,\ldots,M\}$ for transmission over the channel.

When the mapping is performed under the constraint that a waveform transmitted in any time interval depends on one or more previously transmitted waveforms, the modulator is said to have memory. Otherwise, the modulator is called memoryless.

Functional model of passband data transmission system
The digital data transmits over a band-pass channel that can be \textit{linear} or \textit{nonlinear}.

This mode of data transmission relies on the use of a sinusoidal carrier wave modulated by the data stream.

In digital passband transmission, the incoming data stream is modulated onto a carrier (usually sinusoidal) with fixed frequency limits imposed by a band-pass channel of interest.

The modulation process making the transmission possible involves switching (keying) the amplitude, frequency, or phase of a sinusoidal carrier in some fashion in accordance with the incoming data.

There are three basic signaling schemes: amplitude-shift keying (ASK), frequency-shift Keying (FSK), and phase-shift keying (PSK).
4.3 Representation of Digitally Modulated Signals

Illustrative waveforms for the three basic forms of signaling binary information. (a) ASK (b) PSK (c) FSK.
Unlike ASK signals, both PSK and FSK signals have a constant envelope. This property makes PSK and FSK signals impervious to amplitude nonlinearities.

In practice, we find that PSK and FSK signals are preferred to ASK signals for passband data transmission over nonlinear channels.

Digital modulation techniques may be classified into coherent and noncoherent techniques, depending on whether the receiver is equipped with a phase-recovery circuit or not.

The phase-recovery circuit ensures that the oscillator supplying the locally generated carrier wave in the receiver is synchronized (in both frequency and phase) to the oscillator supplying the carrier wave used to originally modulated the incoming data stream in the transmitter.
4.3 Representation of Digitally Modulated Signals

- **M-ary signaling scheme:**
  - For almost all applications, the number of possible signals $M=2^n$.
  - Symbol duration $T=nT_b$, where $T_b$ is the bit duration.
  - We have $M$-ary ASK, $M$-ary PSK, and $M$-ary FSK.
  - We can also combine different methods of modulation into a hybrid form. For example, $M$-ary amplitude-phase keying (APK) and $M$-ary quadrature-amplitude modulation (QAM).
  - $M$-ary PSK and $M$-ary QAM are examples of linear modulation.
  - An $M$-ary PSK signal has a constant envelope, whereas an $M$-ary QAM signal involves changes in the carrier amplitude.
4.3 Representation of Digitally Modulated Signals

- $M$-ary signaling scheme (cont.):
  - $M$-ary PSK can be used to transmit digital data over a **nonlinear band-pass channel**, whereas $M$-ary QAM requires the use of a **linear channel**.
  - $M$-ary PSK, and $M$-ary QAM are commonly used in **coherent systems**.
  - ASK and FSK lend themselves naturally to use in **non-coherent systems** whenever it is impractical to maintain carrier phase synchronization.
  - We can’t have noncoherent PSK.
4.3 Representation of Digitally Modulated Signals

- Power Spectra
  - Given a modulated signal \( s(t) \), we may describe it in terms of its in-phase and quasrature components as:
    \[
    s(t) = s_I(t)\cos(2\pi f_c T) - s_Q(t)\sin(2\pi f_c t)
    \]
    \[
    = \text{Re}\left[\tilde{s}(t)\exp(j2\pi f_c t)\right]
    \]
    \[
    \tilde{s}(t) = s_I(t) + js_Q(t)
    \]
  - The signal \( \tilde{s}(t) \) is the complex envelope (i.e. baseband version) of the modulated (bandpass) signal \( s(t) \). The components \( s_I(t) \), \( s_Q(t) \) and \( \tilde{s}(t) \) are all low-pass signals.
  - Let \( S_B(f) \) denote the power spectral density of the complex envelope \( \tilde{s}(t) \). We refer to \( S_B(f) \) as the baseband power spectral density.
4.3 Representation of Digitally Modulated Signals

- **Power Spectra**
  
  The power spectral density, $S_s(f)$, of the original band-pass signal $s(t)$ is a frequency-shifted version of $S_B(f)$, except for a scaling factor, as shown by
  
  $$S_s(f) = \frac{1}{4} \left[ S_B(f - f_c) + S_B(f + f_c) \right]$$

- **Bandwidth Efficiency**
  
  The bandwidth efficiency is defined as the ratio of the data rate in bits per second to the effectively utilized channel bandwidth:
  
  $$\rho = \frac{R_b}{B} \text{ bits/s/Hz.}$$
4.3.1 Memoryless Modulation Methods

Pulse-amplitude-modulated (PAM) signals

Double-sideband (DSB) signal waveform may be represented as:

\[ s_m(t) = \text{Re} \left[ A_m g(t) e^{j2\pi f_c t} \right] \]

\[ = A_m g(t) \cos 2\pi f_c t, \quad m = 1, 2, ..., M, \quad 0 \leq t \leq T \]

where \( A_m \) denote the set of \( M \) possible amplitudes corresponding to \( M=2^k \) possible \( k \)-bit blocks of symbols.

- The signal amplitudes \( A_m \) take the discrete values:
  \[ A_m = (2m - 1 - M) d, \quad m = 1, 2, ..., M \]
  \[ = -(M-1)d ... (M-1)d \]

- \( 2d \) is the distance between adjacent signal amplitudes.
- \( g(t) \) is a real-valued signal pulse whose shape influences the spectrum of the transmitted signal.
- The symbol rate is \( R/k \), \( T_b=1/R \) is the bit interval, and \( T=k/R=kT_b \) is the symbol interval.
**4.3.1 Memoryless Modulation Methods**

- **Pulse-amplitude-modulated (PAM) signals (cont.)**
  - The $M$ PAM signals have energies:
    \[
    \varepsilon_m = \int_0^T s_m^2(t) \, dt = \frac{1}{2} A_m^2 \int_0^T g^2(t) \, dt = \frac{1}{2} A_m^2 \varepsilon_g
    \]
  - These signals are one-dimensional and are represented by:
    \[
    s_m(t) = s_m f(t)
    \]
  - $f(t)$ is defined as the *unit-energy signal* waveform given as:
    \[
    f(t) = \sqrt{\frac{2}{\varepsilon_g}} g(t) \cos 2\pi f_c t
    \]
    \[
    s_m = A_m \sqrt{\frac{1}{2} \varepsilon_g}, \quad m = 1, 2, \ldots, M
    \]
  - *Digital PAM* is also called *amplitude-shift keying (ASK).*
4.3.1 Memoryless Modulation Methods

- Pulse-amplitude-modulated (PAM) signals (cont.)
  - Signal space diagram for digital PAM signals:

(a) \( M = 2 \)

0 1

00 01 11 10

(b) \( M = 4 \)

000 001 011 010 110 111 101 100

(c) \( M = 8 \)
4.3.1 Memoryless Modulation Methods

Pulse-amplitude-modulated (PAM) signals (cont.)

- **Gray encoding**: The mapping of $k$ information bits to the $M=2^k$ possible signal amplitudes may be done in a number of ways. The preferred assignment is one in which the adjacent signal amplitudes differ by one binary digit.

- The **Euclidean distance** between any pair of signal points is:

$$d_{mn}^{(e)} = \sqrt{(s_m - s_n)^2} = \sqrt{\frac{1}{2} \varepsilon_g |A_m - A_n|} = d \sqrt{2 \varepsilon_g |m - n|}$$

- The **minimum Euclidean distance** between any pair of signals is:

$$d_{\text{min}}^{(e)} = d \sqrt{2 \varepsilon_g}$$
4.3.1 Memoryless Modulation Methods

**Pulse-amplitude-modulated (PAM) signals (cont.)**

- *Single Sideband (SSB) PAM* is represented by:

\[
 s_m(t) = \text{Re} \left\{ A_m \left[ g(t) \pm j \hat{g}(t) \right] e^{j2\pi f_c t} \right\}, \quad m = 1, 2, ..., M
\]

where \( \hat{g}(t) \) is the *Hilbert transform* of \( g(t) \).

- The digital PAM signal is also appropriate for transmission over a channel that does not require carrier modulation and is called *baseband signal*:

\[
 s_m(t) = A_m g(t), \quad m = 1, 2, ..., M
\]

- If \( M=2 \), the signals are called *antipodal* and have the special property that:

\[
 s_1(t) = -s_2(t)
\]
4.3.1 Memoryless Modulation Methods

- Pulse-amplitude-modulated (PAM) signals (cont.)
  - Four-amplitude level baseband and band-pass PAM signals
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals (Binary Phase-Shift Keying)
  - In a coherent binary PSK system, the pair of signals $s_1(t)$ and $s_2(t)$ used to represent binary symbols 1 and 0, respectively is defined by

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$$

where $0 \leq t \leq T_b$ and $E_b$ is the transmitted signal energy per bit.

- A pair of sinusoidal waves that differ only in a relative phase-shift of 180 degrees are referred to as antipodal signals.
4.3.1 Memoryless Modulation Methods

Phase-modulated signals (Binary Phase-Shift Keying)

- To ensure that each transmitted bit contains an integral number of cycles of the carrier wave, the carrier frequency $f_c$ is chosen equal to $n_c/T_b$ for some fixed integer $n_c$.
- In the case of binary PSK, there is only one basis function of unit energy:
  $$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t), \quad 0 \leq t < T_b$$
  $$s_1(t) = \sqrt{E_b} \phi_1(t), \quad s_2(t) = -\sqrt{E_b} \phi_1(t), \quad 0 \leq t < T_b$$
- The coordinates of the message points are:
  $$s_{11} = \int_0^{T_b} s_1(t) \phi_1(t) \, dt = +\sqrt{E_b} \quad s_{21} = \int_0^{T_b} s_2(t) \phi_1(t) \, dt = -\sqrt{E_b}$$
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals (Binary Phase-Shift Keying)
**4.3.1 Memoryless Modulation Methods**

- **Phase-modulated signals (Binary Phase-Shift Keying)**
  - The decision region associated with symbol 1 or signal $s_1(t)$ is described by $Z_1: 0 < x_1 < \infty$.
  - The conditional probability density function of random variable $X_1$, given that symbol 0 (i.e., signal $s_2(t)$) was transmitted, is defined by:

$$f_{x_1}(x_1 \mid 0) = \frac{1}{\sqrt{\pi N_0}} \exp \left[ - \frac{1}{N_0} (x_1 - s_{21})^2 \right]$$

$$= \frac{1}{\sqrt{\pi N_0}} \exp \left[ - \frac{1}{N_0} (x_1 + \sqrt{E_b})^2 \right]$$
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals (Binary Phase-Shift Keying)
  - The conditional probability of the receiver deciding in favor of symbol 1, given that symbol 0 was transmitted, is therefore
  \[
  p_{10} = \int_{0}^{\infty} f_{x_1}(x_1 | 0) \, dx_1 = \frac{1}{\sqrt{\pi N_0}} \int_{0}^{\infty} \exp \left[ -\frac{1}{N_0} (x_1 + \sqrt{E_b})^2 \right] \, dx_1
  \]
  - Let \( z = \frac{1}{\sqrt{N_0}} (x_1 + \sqrt{E_b}) \)
  \[
  p_{10} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b / N_0}}^{\infty} \exp \left( -z^2 \right) \, dt = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right)
  \]
  - The average probability of symbol (bit) error is
  \[
  p_e = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right)
  \]
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals (Binary Phase-Shift Keying)
  - Symbol Shaping Function
    \[ g(t) = \begin{cases} \sqrt{\frac{2E_b}{N_0}}, & 0 \leq t \leq T_b \\ 0, & \text{otherwise} \end{cases} \]
  - Power Spectra
    \[ S_b(f) = \frac{2E_b \sin^2(\pi T_b f)}{(\pi T_b f)^2} = 2E_b \text{sinc}^2(T_b f) \]

- Symbol Shaping Function
- Power Spectra
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals (Quadriphase-Shift Keying)
  - Quadriphase-Shift Keying (QPSK)

\[
s_i(t) = \begin{cases} 
\sqrt{\frac{2E}{T}} \cos \left[ 2\pi f_c t + (2i-1) \frac{\pi}{4} \right], & 0 \leq t \leq T \\
0, & \text{elsewhere}
\end{cases}
\]

where \( i = 1, 2, 3, 4; \) \( E \) is the transmitted signal energy per symbol, and \( T \) is the symbol duration.

- Equivalently

\[
s_i(t) = \sqrt{\frac{2E}{T}} \cos \left[ (2i-1) \frac{\pi}{4} \right] \cos (2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin \left[ (2i-1) \frac{\pi}{4} \right] \sin (2\pi f_c t)
\]
**4.3.1 Memoryless Modulation Methods**

- **Phase-modulated signals (Quadriphase-Shift Keying)**
  - Defined a pair of quadrature carriers:
    
    \[
    \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \leq t \leq T
    \]
    
    \[
    \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad 0 \leq t \leq T
    \]
  
  - There are four message points, and the associated signal vectors are defined by
    
    \[
    s_i(t) = \begin{bmatrix}
    \sqrt{E} \cos \left( (2i - 1) \frac{\pi}{4} \right) \\
    -\sqrt{E} \sin \left( (2i - 1) \frac{\pi}{4} \right)
    \end{bmatrix}, \quad i = 1, 2, 3, 4
    \]
Phase-modulated signals (Quadriphase-Shift Keying)
- Each possible value of the phase corresponds to a unique dibit.
- For example, we may choose the Gray coding.

<table>
<thead>
<tr>
<th>Gray-encoded Input Dibit</th>
<th>Phase of QPSK Signal (radians)</th>
<th>Coordinates of Message Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$\pi/4$</td>
<td>$+\sqrt{E/2}$, $-\sqrt{E/2}$</td>
</tr>
<tr>
<td>00</td>
<td>$3\pi/4$</td>
<td>$-\sqrt{E/2}$, $-\sqrt{E/2}$</td>
</tr>
<tr>
<td>01</td>
<td>$5\pi/4$</td>
<td>$-\sqrt{E/2}$, $+\sqrt{E/2}$</td>
</tr>
<tr>
<td>11</td>
<td>$7\pi/4$</td>
<td>$+\sqrt{E/2}$, $+\sqrt{E/2}$</td>
</tr>
</tbody>
</table>
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals (Quadriphase-Shift Keying)
- Signal space diagram of coherent QPSK system
4.3.1 Memoryless Modulation Methods

Phase-modulated signals (Quadriphase-Shift Keying)

Input binary sequence

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dibit 01</td>
<td>Dibit 10</td>
</tr>
</tbody>
</table>

(a)

Odd-numbered sequence
Polarity of coefficient $s_{\ell 1}$

1

1

O

(b)

Even-numbered sequence
Polarity of coefficient $s_{\ell 2}$

1

0

0

0

(c)

$s_{\ell 1}\phi_{\ell 1}(t)$

$s_{\ell 2}\phi_{\ell 2}(t)$

(d)

$s(t)$

70
Phase-modulated signals (Quadriphase-Shift Keying)

Error Probability of QPSK

In a coherent QPSK system the received signal \( x(t) \) is defined by
\[
x(t) = s_i(t) + w(t), \quad \left\{ \begin{array}{l}
0 \leq t \leq T \\
i = 1, 2, 3, 4
\end{array} \right.
\]

The observation vector \( \mathbf{x} \) has two elements, \( x_1 \) and \( x_2 \)
\[
x_1 = \int_0^T x(t) \phi_1(t) \ dt = \sqrt{E} \cos \left( (2i - 1) \frac{\pi}{4} \right) + w_1 = \pm \sqrt{\frac{E}{2}} + w_1
\]
\[
x_2 = \int_0^T x(t) \phi_2(t) \ dt = -\sqrt{E} \sin \left( (2i - 1) \frac{\pi}{4} \right) + w_2 = \mp \sqrt{\frac{E}{2}} + w_2
\]
Phase-modulated signals (Quadriphase-Shift Keying)

Error Probability of QPSK (cont.)

Note that a coherent QPSK system is equivalent to two coherent binary PSK systems working in parallel and using two carriers that are in phase quadrature.

- The signal energy per bit is $E/2$.
- The noise spectral density is $N_0/2$.

In each channel of the coherent QPSK system:

$$P' = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E}{2N_0}} \right) = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E}{2N_0}} \right)$$
4.3.1 Memoryless Modulation Methods

Phase-modulated signals (Quadriphase-Shift Keying)

Error Probability of QPSK (cont.)

The average probability of a correct decision:

\[ P_c = (1 - P')^2 = \left[ 1 - \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E}{2N_0}} \right) \right]^2 \]

\[ = 1 - \text{erfc} \left( \sqrt{\frac{E}{2N_0}} \right) + \frac{1}{4} \text{erfc}^2 \left( \sqrt{\frac{E}{2N_0}} \right) \]

The average probability of symbol error rate for coherent QPSK is

\[ P_e = 1 - P_c = \text{erfc} \left( \sqrt{\frac{E}{2N_0}} \right) - \frac{1}{4} \text{erfc}^2 \left( \sqrt{\frac{E}{2N_0}} \right) \]
Phase-modulated signals (Quadriphase-Shift Keying)

- Error Probability of QPSK (cont.)
  
  Since there are two bits per symbol, we have $E = 2E_b$.
  
  With Gray encoding used for the incoming symbols, the bit error rate of QPSK is given by

  $$BER = \frac{1}{2} \erfc \left( \sqrt{\frac{E_b}{N_0}} \right)$$

- A coherent QPSK system achieves the same average probability of bit error as a coherent binary PSK system for the same bit rate and the same $E_b/N_0$, but uses only half the channel bandwidth.
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals (Quadriphase-Shift Keying)
  - Generation of coherent QPSK signals

\[
\phi_1(t) = \sqrt{2/\Gamma} \cos(2\pi f_c t)
\]

\[
\phi_2(t) = \sqrt{2/\Gamma} \sin(2\pi f_c t)
\]
Phase-modulated signals (Quadriphase-Shift Keying)

Power spectra of QPSK signals

\[ S_B(f) = 2E \ \text{sinc}^2(Tf) = 4E_b \ \text{sinc}^2(2T_b f) \]
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals ($M$-ary PSK)
  - The $M$ signal waveforms are represented as:

  $$s_m(t) = \text{Re} \left[ g(t) e^{j2\pi (m-1)/M} e^{j2\pi f_c t} \right], \quad m = 1, 2, ..., M, \quad 0 \leq t \leq T$$

  $$= g(t) \cos \left[ 2\pi f_c t + \frac{2\pi}{M} (m-1) \right]$$

  $$= g(t) \cos \frac{2\pi}{M} (m-1) \cos 2\pi f_c t - g(t) \sin \frac{2\pi}{M} (m-1) \sin 2\pi f_c t$$

- **Digital phase modulation** is usually called *phase-shift keying* (PSK).
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals ($M$-ary PSK)
  - Signal space diagram for octaphase shift keying (i.e., $M=8$)
**4.3.1 Memoryless Modulation Methods**

- **Phase-modulated signals** (\(M\)-ary PSK)
  - The signal waveforms have equal energy:
    \[
    \varepsilon = \int_0^T s_m^2(t)\,dt = \frac{1}{2} \int_0^T g^2(t)\,dt = \frac{1}{2} \varepsilon_g
    \]
  - The signal waveforms may be represented as a linear combination of two orthonormal signal waveforms:
    \[
    s_m(t) = s_{m1}f_1(t) + s_{m2}f_2(t)
    \]
    \[
    f_1(t) = \sqrt{\frac{2}{\varepsilon_g}} g(t)\cos 2\pi f_c t \quad \text{and} \quad f_2(t) = -\sqrt{\frac{2}{\varepsilon_g}} g(t)\sin 2\pi f_c t
    \]
  - The two-dimensional vectors \(s_m = [s_{m1}, s_{m2}]\) are given by:
    \[
    s_m = \begin{bmatrix}
    \sqrt{\frac{2}{\varepsilon_g}} \cos \frac{2\pi}{M} (m-1) \\
    \sqrt{\frac{2}{\varepsilon_g}} \sin \frac{2\pi}{M} (m-1)
    \end{bmatrix}, \quad m = 1, 2, \ldots, M
    \]
4.3.1 Memoryless Modulation Methods

- Phase-modulated signals ($M$-ary PSK)
  - Signal space diagram illustrating the application of the union bound for octaphase-shift keying
Phase-modulated signals ($M$-ary PSK)

The Euclidean distance between signal points is:

$$d_{mn}^{(e)} = \| s_m - s_n \|$$

$$= \sqrt{(s_{m1} - s_{n1})^2 + (s_{m2} - s_{n2})^2} = \left\{ \varepsilon_g \left[ 1 - \cos \frac{2\pi}{M} (m - n) \right] \right\}^{1/2}$$

The minimum Euclidean distance corresponds to the case in which $|m-n|=1$, i.e., adjacent signal phases.

$$d_{\text{min}}^{(e)} = \sqrt{\varepsilon_g \left( 1 - \cos \frac{2\pi}{M} \right)}$$

Average probability of symbol error for coherent $M$-ary PSK:

$$P_e \approx \text{erfc} \left( \sqrt{\frac{E}{N_0} \sin \left( \frac{\pi}{M} \right)} \right)$$
Phase-modulated signals (\(M\)-ary PSK)

Power Spectra of \(M\)-ary PSK Signals

\[
S_B(f) = 2E \text{sinc}^2(Tf) \\
= 2E_b \log_2 M \text{sinc}^2(T_b f \log_2 M)
\]
Phase-modulated signals (Offset QPSK)

- The carrier phase changes by ±180 degrees whenever both the in-phase and quadrature components of the QPSK signal changes sign.
- This can result in problems for power amplifiers.
- The problem may be reduced by using offset QPSK.
- In offset QPSK, the bit stream responsible for generating the quadrature component is delayed (i.e. offset) by half a symbol interval with respect to the bit stream responsible for generating the in-phase component.

\[
\phi_i(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \leq t \leq T
\]
Phase-modulated signals (Offset QPSK)

The two basis functions of offset QPSK are defined by

\[ \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \leq t \leq T \]

\[ \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad \frac{T}{2} \leq t \leq \frac{3T}{2} \]

The phase transitions likely to occur in offset QPSK are confined to ±90 degrees.

However, ±90 degrees phase transitions in offset QPSK occur twice as frequently.
Phase-modulated signals (Offset QPSK)

- Amplitude fluctuations in offset QPSK due to filtering have a smaller amplitude than in the case of QPSK.
- The offset QPSK has exactly the same probability of symbol error in an AWGN channel as QPSK.
- The reason for the equivalence is that the statistical independence of the in-phase and quadrature components applies to both QPSK and offset QPSK.
4.3.1 Memoryless Modulation Methods

Phase-modulated signals (\(\pi/4\)-Shifted QPSK)

- The carrier phase used for the transmission of successive symbols is alternately picked from one of the two QPSK constellations in the following figure and then the other.
Phased-modulated signals (π/4-Shifted QPSK)

It follows that a π/4-shifted QPSK signal may reside in any one of eight possible phase states:

<table>
<thead>
<tr>
<th>Gray-Encoded Input Dibit</th>
<th>Phase Change, Δθ (radians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>π/4</td>
</tr>
<tr>
<td>01</td>
<td>3π/4</td>
</tr>
<tr>
<td>11</td>
<td>−3π/4</td>
</tr>
<tr>
<td>10</td>
<td>−π/4</td>
</tr>
</tbody>
</table>
Phase-modulated signals (π/4-Shifted QPSK)

- Attractive features of the π/4-shifted QPSK scheme
  - The phase transitions from one symbol to the next are restricted to ±π/4 and ±3π/4.
  - Envelope variations due to filtering are significantly reduced.
  - π/4-shifted QPSK signals can be noncoherently detected, thereby considerably simplifying the receiver design.
  - Like QPSK signals, π/4-shifted QPSK can be differently encoded, in which case we should really speak of π/4-shifted DQPSK.
4.3.1 Memoryless Modulation Methods

- **Dual-Carrier Modulation (DCM)**
  - Adopted in Multi-band OFDM (Ultra Wideband)
  - The coded and interleaved binary serial input data, \( b[i] \) where \( i = 0, 1, 2, \cdots \), shall be divided into groups of 200 bits and converted into 100 complex numbers using a technique called dual-carrier modulation.
  - The conversion shall be performed as follows:
    1. The 200 coded bits are grouped into 50 groups of 4 bits. Each group is represented as \((b[g(k)], \ b[g(k)+1], \ b[g(k)+50], \ b[g(k)+51])\), where \( k \in [0, 49] \) and
      \[
      g(k) = \begin{cases} 
      2k & k \in [0, 24] \\
      2k + 50 & k \in [25, 49] 
      \end{cases}
      \]
4.3.1 Memoryless Modulation Methods

- **Dual-Carrier Modulation (DCM)**

2. Each group of 4 bits \((b[g(k)], b[g(k) + 1], b[g(k) + 50], b[g(k) + 51])\) shall be mapped onto a four-dimensional constellation, and converted into two complex numbers \((d[k], d[k + 50])\).

3. The complex numbers shall be normalized using a normalization factor \(K_{MOD}\).

- The normalization factor \(K_{MOD} = 10^{-1/2}\) is used for the dual-carrier modulation.

- An approximate value of the normalization factor may be used, as long as the device conforms to the modulation accuracy requirements.
4.3.1 Memoryless Modulation Methods

- Dual-Carrier Modulation (DCM)

![Diagram showing Dual-Carrier Modulation (DCM)]
### 4.3.1 Memoryless Modulation Methods

#### Dual-Carrier Modulation (DCM) Encoding Table

<table>
<thead>
<tr>
<th>Input Bit ( (b_g(k), b_g(k+1), b_g(k+50), b_g(k+51)) )</th>
<th>( d[k] ) I-out</th>
<th>( d[k] ) Q-out</th>
<th>( d[k+50] ) I-out</th>
<th>( d[k+50] ) Q-out</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>-3</td>
<td>-3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0001</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>0010</td>
<td>-3</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>0011</td>
<td>-3</td>
<td>3</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>0100</td>
<td>-1</td>
<td>-3</td>
<td>-3</td>
<td>1</td>
</tr>
<tr>
<td>0101</td>
<td>-1</td>
<td>-1</td>
<td>-3</td>
<td>-3</td>
</tr>
<tr>
<td>0110</td>
<td>-1</td>
<td>1</td>
<td>-3</td>
<td>3</td>
</tr>
<tr>
<td>0111</td>
<td>-1</td>
<td>3</td>
<td>-3</td>
<td>-1</td>
</tr>
<tr>
<td>1000</td>
<td>1</td>
<td>-3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1001</td>
<td>1</td>
<td>-1</td>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>1010</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1011</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>1100</td>
<td>3</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1101</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td>1110</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>1111</td>
<td>3</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>
4.3.1 Memoryless Modulation Methods

- Quadrature amplitude modulation (QAM)
  - *Quadrature PAM* or *QAM*: The bandwidth efficiency of PAM/SSB can also be obtained by simultaneously impressing two separate $k$-bit symbols from the information sequence \( \{a_n\} \) on two quadrature carriers \( \cos 2\pi f_c t \) and \( \sin 2\pi f_c t \).
  - The signal waveforms may be expressed as:
    \[
    s_m(t) = \text{Re} \left[ \left( A_{mc} + jA_{ms} \right) g(t) e^{j2\pi f_c t} \right], \quad m = 1, 2, \ldots, M, \quad 0 \leq t \leq T
    \]
    \[
    = A_{mc} g(t) \cos 2\pi f_c t - A_{ms} g(t) \sin 2\pi f_c t
    \]
    \[
    s_m(t) = \text{Re} \left[ V_m e^{j\theta_m} g(t) e^{j2\pi f_c t} \right] = V_m g(t) \cos (2\pi f_c t + \theta_m)
    \]
    \[
    V_m = \sqrt{A_{mc}^2 + A_{ms}^2} \quad \text{and} \quad \theta_m = \tan^{-1} \left( \frac{A_{ms}}{A_{mc}} \right)
    \]
4.3.1 Memoryless Modulation Methods

- Quadrature amplitude modulation (QAM) (cont.)
  - We may select a combination of $M_1$-level PAM and $M_2$-phase PSK to construct an $M=M_1M_2$ combined PAM-PSK signal constellation.
4.3.1 Memoryless Modulation Methods

Quadrature amplitude modulation (QAM) (cont.)

- As in the case of PSK signals, the QAM signal waveforms may be represented as a linear combination of two orthonormal signal waveforms $f_1(t)$ and $f_2(t)$:

$$s_m(t) = s_{m1}f_1(t) + s_{m2}f_2(t)$$

$$f_1(t) = \sqrt{\frac{2}{\varepsilon_g}}g(t)\cos 2\pi f_c t \quad \text{and} \quad f_2(t) = -\sqrt{\frac{2}{\varepsilon_g}}g(t)\sin 2\pi f_c t$$

$$s_m = \begin{bmatrix} s_{m1} & s_{m2} \end{bmatrix} = \begin{bmatrix} A_{mc}\sqrt{\frac{1}{2}\varepsilon_g} & A_{ms}\sqrt{\frac{1}{2}\varepsilon_g} \end{bmatrix}$$

- The Euclidean distance between any pair of signal vectors is:

$$d_{mn}^{(e)} = \|s_m - s_n\| = \sqrt{\frac{1}{2}\varepsilon_g \left[ (A_{mc} - A_{nc})^2 + (A_{ms} - A_{ns})^2 \right]}$$


4.3.1 Memoryless Modulation Methods

- Quadrature amplitude modulation (QAM) (cont.)
  - Several signal space diagrams for *rectangular QAM*:

\[ d^{(e)}_{\text{min}} = d \sqrt{2\varepsilon_g} \]
Multidimensional signals:

- We may use either the time domain or the frequency domain or both in order to increase the number of dimensions.

- Subdivision of time and frequency axes into distinct slots:
4.3.1 Memoryless Modulation Methods

- Orthogonal multidimensional signals
  - Consider the construction of $M$ equal-energy orthogonal signal waveforms that differ in frequency:

  $$
s_m(t) = \text{Re} \left[ s_{lm}(t) e^{j2\pi f_c t} \right], \quad m = 1, 2, \ldots, M, \quad 0 \leq t \leq T
  = \frac{\sqrt{2\varepsilon}}{T} \cos \left[ 2\pi f_c t + 2\pi m\Delta f t \right]
  \quad s_{lm}(t) = \frac{\sqrt{2\varepsilon}}{T} e^{j2\pi m\Delta f t}, \quad m = 1, 2, \ldots, M, \quad 0 \leq t \leq T
  $$

- This type of frequency modulation is called *frequency-shift keying* (FSK).
4.3.1 Memoryless Modulation Methods

Orthogonal multidimensional signals (cont.)

These waveforms have equal cross-correlation coefficients:

\[
\rho_{km} = \frac{2\varepsilon/T}{2\varepsilon} \int_0^T e^{j2\pi(m-k)\Delta f} dt = \frac{\sin \pi T (m-k) \Delta f}{\pi T (m-k) \Delta f} e^{j\pi T (m-k) \Delta f}
\]

4.2-45

\[
\rho_r \equiv \Re(\rho_{km}) = \frac{\sin[\pi T (m-k) \Delta f]}{\pi T (m-k) \Delta f} \cos[\pi T (m-k) \Delta f]
\]

\[
= \frac{\sin[2\pi T (m-k) \Delta f]}{2\pi T (m-k) \Delta f}
\]

Note that \(\Re(\rho_{km})=0\) when \(\Delta f=1/2T\) and \(m \neq k\).

| \(\rho_{km}||=0\) for multiple of \(1/T\).
4.3.1 Memoryless Modulation Methods

- Orthogonal multidimensional signals (cont.)
  - Cross-correlation coefficient as a function of frequency separation for FSK signals:

---

In each segment of length $T$ may be used to simulate the $N$-dimensional signal vector by modulating the $N$-dimensional vector independently by the corresponding $N$-dimension signal vector. Alternatively, a $N$-dimensional vector is transmitted in $NT$ frequency slots each of width $\Delta f / \Delta t$. If $N$-dimensional vectors are used in each frequency slot, no cross-talk occurs in the $N$-dimensional signals.
Orthogonal multidimensional signals (cont.)

For $\Delta f = 1/2T$, the $M$-FSK signals are equivalent to the $N$-dimensional vectors:

$$s_1 = \begin{bmatrix} \sqrt{\epsilon} & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

$$s_2 = \begin{bmatrix} 0 & \sqrt{\epsilon} & 0 & \ldots & 0 & 0 \end{bmatrix}$$

$$\vdots \quad \vdots$$

$$s_N = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & \sqrt{\epsilon} \end{bmatrix}$$

The distance between pairs of signals is:

$$d_{km}^{(e)} = \sqrt{2\epsilon}, \quad \text{for all } m, k$$
4.3.1 Memoryless Modulation Methods

- Orthogonal multidimensional signals (cont.)

- Orthogonal signals for $M=N=3$ and $M=N=2$.
Binary Frequency-Shift Keying

In a binary FSK system, symbols 1 and 0 are distinguished from each other by transmitting one of two sinusoidal waves that differ in frequency by a fixed amount.

\[ s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t), & 0 \leq t \leq T_b \\ 0, & \text{elsewhere} \end{cases} \]

The transmitted frequency is

\[ f_i = \frac{n_c + i}{T_b} \quad \text{for some fixed integer } n_c \text{ and } i = 1, 2 \]
4.3.1 Memoryless Modulation Methods

- **Binary Frequency-Shift Keying**
  - The FSK signal is a continuous phase signal in the sense that phase continuity is always maintained, including the inter-bit switching times.
  - This form of digital modulation is an example of *continuous-phase frequency-shift keying* (CPFSK).
  - The signal $s_1(t)$ and $s_2(t)$ are orthogonal

$$
\phi_i(t) = \begin{cases} 
\sqrt{\frac{2}{T_b}} \cos(2 \pi f_i t), & 0 \leq t \leq T_b \\
0, & \text{elsewhere}
\end{cases}
$$
4.3.1 Memoryless Modulation Methods

**Binary Frequency-Shift Keying**

\[
s_{ij} = \int_{0}^{T_b} s_i(t) \phi_j(t) dt
\]

\[
= \int_{0}^{T_b} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t) dt
\]

\[
= \begin{cases} 
\sqrt{E_b}, & i = j \\
0, & i \neq j
\end{cases}
\]

\[
s_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} 
\]

\[
s_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}
\]
4.3.1 Memoryless Modulation Methods

- Binary Frequency-Shift Keying
- Signal space diagram for binary FSK system
4.3.1 Memoryless Modulation Methods

- Binary Frequency-Shift Keying
  - Error probability of binary FSK
    - The observation vector \( \mathbf{x} \) has two elements:
      \[
      x_1 = \int_{0}^{T_b} x(t) \phi_1(t) \, dt
      \]
      \[
      x_2 = \int_{0}^{T_b} x(t) \phi_2(t) \, dt
      \]
    - The observation space is partitioned into two decision regions, labeled \( Z_1 \) and \( Z_2 \).
    - Define a new Gaussian random variable \( Y \) whose sample value \( y \) is equal to the difference between \( x_1 \) and \( x_2 \).
      \[
      y = x_1 - x_2
      \]
4.3.1 Memoryless Modulation Methods

- **Binary Frequency-Shift Keying**

- **Error probability of binary FSK (cont.)**

\[
E[Y|1] = E[X_1|1] - E[X_2|1] = +\sqrt{E_b}
\]

\[
E[Y|0] = E[X_1|0] - E[X_2|0] = -\sqrt{E_b}
\]

\[
\text{var}[Y] = \text{var}[X_1] + \text{var}[X_2] = N_0
\]

\[
f_Y(y|0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{(y + \sqrt{E_b})^2}{2N_0}\right]
\]
4.3.1 Memoryless Modulation Methods

- **Binary Frequency-Shift Keying**
- **Error probability of binary FSK (cont.)**

\[ p_{10} = P\left( y > 0 \mid \text{symbol 0 was sent} \right) = \int_0^\infty f_y(y|0) \, dy \]

\[ = \frac{1}{\sqrt{2\pi N_0}} \int_0^\infty \exp\left[-\frac{(y + \sqrt{E_b})^2}{2 N_0}\right] \, dy \]

\[ = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b}/2N_0}^\infty \exp(-z^2) \, dz = \frac{1}{2} \text{erfc}\left(\frac{\sqrt{E_b}}{2N_0}\right) \]

\[ P_e = \frac{1}{2} \text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \]
4.3.1 Memoryless Modulation Methods

- Binary Frequency-Shift Keying
- Generation of Coherent Binary FSK Signals

\[ \phi_1(t) = \sqrt{2/T_b} \cos(2\pi f_1 t) \]

\[ \phi_2(t) = \sqrt{2/T_b} \cos(2\pi f_2 t) \]

Binary data sequence -> On-off level encoder -> Inverter

- \( m(t) \)
- Binary FSK signal \( s(t) \)
4.3.1 Memoryless Modulation Methods

**Binary Frequency-Shift Keying**

- Power spectra of binary FSK signals
  - The symbol shaping function is defined by
    \[
    g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \sin\left(\pm \frac{\pi t}{T_b}\right), & 0 \leq t \leq T_b \\ 0, & \text{elsewhere} \end{cases}
    \]
  - The energy spectral density of this symbol shaping function
    \[
    \Psi_g(f) = \frac{8E_bT_b \cos(\pi T_b f)^2}{\pi^2 \left(4T_b^2 f^2 - 1\right)^2}
    \]
  - The baseband power spectral density is given by
    \[
    S_B(f) = \frac{E_b}{2T_b} \left[ \delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right) \right] + \frac{8E_bT_b \cos(\pi T_b f)^2}{\pi^2 \left(4T_b^2 f^2 - 1\right)^2}
    \]
4.3.1 Memoryless Modulation Methods

- **M-ary Frequency-Shift Keying**
  - The transmitted signals are defined by
    \[
    s_i(t) = \sqrt{\frac{2E}{T}} \cos \left( \frac{\pi}{T} \left( n_c + i \right) t \right), \quad 0 \leq t \leq T
    \]
  - The carrier frequency \( f_c = n_c / 2T \) for some fixed integer \( n_c \).
  - Since the individual signal frequencies are separated by \( 1/2T \) Hz, the signals are orthogonal:
    \[
    \int_0^T s_i(t)s_j(t)dt = 0, \quad i \neq j
    \]
4.3.1 Memoryless Modulation Methods

- **M-ary Frequency-Shift Keying**
  - A complete orthonormal set of basis function is given by
    \[
    \phi_i(t) = \frac{1}{\sqrt{E}} s_i(t), \quad 0 \leq t \leq T, \quad i = 1, 2, \ldots, M
    \]
  - The M-ary FSK is described by an M-dimensional signal space diagram.
  - The upper bound on the average probability of symbol error for M-ary FSK is given by:
    \[
    p_e \leq \frac{1}{2} (M - 1) \text{erfc} \left( \sqrt{\frac{E}{2N_0}} \right)
    \]
4.3.1 Memoryless Modulation Methods

- \( M \)-ary Frequency-Shift Keying
  - Power spectra of \( M \)-ary FSK signals

*Figure 6.36*  Power spectra of \( M \)-ary PSK signals for \( M = 2, 4, 8 \).
Biorthogonal signals

A set of $M$ biorthogonal signals can be constructed from $M/2$ orthogonal signals by simply including the negatives of the orthogonal signals.

The correlation between any pair of waveforms is either $\rho_r = -1$ or 0.

Signal space diagrams for $M=4$ and $M=6$ biorthogonal signals.
4.3.1 Memoryless Modulation Methods

- **Simplex signals**
  - For a set of $M$ orthogonal waveforms $\{s_m(t)\}$ or their vector representation $\{s_m\}$ with mean of:
    $$s = \frac{1}{M} \sum_{m=1}^{M} s_m$$

  *Simplex signals* are obtained by translating the origin of the $m$ orthogonal signals to the point $s$.
  $$s_m' = s_m - s, \quad m = 1, 2, \ldots, M$$

  - The energy per waveform is:
    $$\|s_m'\|^2 = \|s_m - s\|^2 = \varepsilon - \frac{2}{M} \varepsilon + \frac{1}{M} \varepsilon = \varepsilon \left(1 - \frac{1}{M}\right)$$
Simplex signals (cont.)

- The cross correlation of any pair of signals is (4.2-27):
\[ \text{Re}(\rho_{mn}) = \frac{s'_m \cdot s'_n}{\|s'_m\| \|s'_n\|} = \frac{-1/M}{1 - 1/M} \]
\[ = - \frac{1}{M - 1} \]

- The set of simplex waveforms is *equally correlated* and requires less energy, by the factor $1 - 1/M$, than the set of orthogonal waveforms.
4.3.1 Memoryless Modulation Methods

Signal waveforms from binary codes

A set of $M$ signaling waveforms can be generated from a set of $M$ binary code words of the form:

$$C_m = [c_{m1} \ c_{m2} \ \cdots \ c_{mN}], \ m = 1, 2, \ldots, M, \ c_{mj} = 0 \ or \ 1.$$  

Each component of a code word is mapped into an elementary binary PSK waveform:

$$c_{mj} = 1 \Rightarrow s_{mj}(t) = \sqrt{\frac{2\epsilon}{T_c}} \cos 2\pi f_c t, \ 0 \leq t \leq T_c$$

$$c_{mj} = 0 \Rightarrow s_{mj}(t) = -\sqrt{\frac{2\epsilon}{T_c}} \cos 2\pi f_c t, \ 0 \leq t \leq T_c$$

where $T_c = T/N$ and $\epsilon_c = \epsilon /N$. 
4.3.1 Memoryless Modulation Methods

Signal waveforms from binary codes (cont.)

- The $M$ code words $\{C_m\}$ are mapped into a set of $M$ waveforms $\{s_m(t)\}$.
- The waveforms can be represented in vector form as:
  
  $s_m = [s_{m1} \ s_{m2} \ \cdots \ s_{mN}]$, \quad $m = 1, 2, \ldots, M$, \quad where \ $s_{mj} = \pm \sqrt{\varepsilon / N}$.

- $N$ is called the block length of the code and is also the dimension of the $M$ waveforms.
Signal waveforms from binary codes (cont.)

- Each of the $M$ waveforms has energy $\varepsilon$.
- Any adjacent signal points have a cross-correlation coefficient:
  \[
  \rho_r = \frac{\varepsilon(1 - 2/N)}{\varepsilon} = \frac{N - 2}{N}
  \]
- The corresponding distance is:
  \[
  d^{(e)} = \sqrt{2\varepsilon(1 - \rho_r)} = \sqrt{4\varepsilon/N}
  \]
4.3.2 Linear Modulation with Memory

- There are some modulation signals with dependence between the signals transmitted in successive symbol intervals.
- This signal dependence is usually introduced for the purpose of shaping the spectrum of the transmitted signal so that it matches the spectral characteristics of the channel.

Examples of baseband signals and the corresponding data sequence.
NRZ: the binary information digit 1 is represented by a rectangular pulse of polarity $A$ and the binary digit 0 is represented by a rectangular pulse of polarity $-A$.

The NRZ modulation is memoryless and is equivalent to a binary PAM or a binary PSK signal in a carrier-modulated system.

NRZI: the signal is different from the NRZ signal in that transitions from one amplitude level to another occur only when a 1 is transmitted.

This type of signal encoding is called \textit{differential encoding}.

The encoding operation is described by the relation:

$$b_k = a_k \oplus b_{k-1}$$

where \{a_k\} is the binary information sequence into the encoder, \{b_k\} is the output sequence of the encoder.
4.3.2 Linear Modulation with Memory

NRZI (cont.)

- When $b_k=1$, the transmitted waveform is a rectangular pulse of amplitude $A$, and when $b_k=0$, the transmitted waveform is a rectangular pulse of amplitude $-A$.
- The differential encoding operation introduces memory in the signal.
- The combination of the encoder and the modulator operations may be represented by a state diagram (Markov chain):

- The input bit $0$ or $1$ affects the state transition from $S_0$ to $S_1$ or vice versa.

![State Diagram](image-url)
NRZI (cont.)

The state diagram may be described by two transition matrices corresponding to the two possible input bits \( \{0,1\} \).

When \( a_k=0 \), the encoder stays in the same state and the state transition matrix for a zero is:

\[
T_1 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

where \( t_{ij} = 1 \) if \( a_k \) results in a transition from state \( i \) to state \( j \), \( i=1,2 \), and \( j=1,2 \); otherwise \( t_{ij} = 0 \).

The state transition matrix for \( a_k=1 \) is:

\[
T_2 = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
NRZI (cont.)

The *trellis diagram* for the NRZI signal:

*Delay modulation* is equivalent to encoding the data sequence by a run-length-limited code called a *Miller code* and using NRZI to transmit the encoded data (will be shown in Chapter 9).
Delay modulation (cont.):

The signal of delay modulation may be described by a state diagram that has four states:

- There are two elementary waveforms $s_1(t)$ and $s_2(t)$ and their negatives $-s_1(t)$ and $-s_2(t)$, which are used for transmitting the binary information.
**4.3.2 Linear Modulation with Memory**

- **Delay modulation**
  - The state transition matrices can be obtained from the figure in previous page:

  When \( a_k = 0 \):
  \[
  T_1 = \begin{bmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  \end{bmatrix}
  \]

  \( s_3 \rightarrow s_1 \)

  \( s_4 \rightarrow s_1 \)

  \( s_1 \rightarrow s_4 \)

  \( s_2 \rightarrow s_4 \)

  When \( a_k = 1 \):
  \[
  T_2 = \begin{bmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  \end{bmatrix}
  \]

  \( s_1 \rightarrow s_2 \)

  \( s_3 \rightarrow s_2 \)

  \( s_2 \rightarrow s_3 \)

  \( s_4 \rightarrow s_3 \)
Modulation techniques with memory such as NRZI and Miller coding are generally characterized by a $K$-state Markov chain with \textit{stationary state probabilities} \{\(p_i, i=1,2,\ldots,K\}\) and \textit{transition probabilities} \{\(p_{ij}, i,j=1,2,\ldots,K\}\). Associated with each transition is a signal waveform \(s_j(t), j=1,2,\ldots,K\).

\textit{Transition probability matrix} (\(P\)) can be arranged in matrix form as:

\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1K} \\
p_{21} & p_{22} & \cdots & p_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
p_{K1} & p_{K2} & \cdots & p_{KK}
\end{bmatrix}
\]
4.3.2 Linear Modulation with Memory

- The transition probability matrix can be obtained from the transition matrices \( \{T_i\} \) and the corresponding probabilities of occurrence of the input bits:

\[
P = \sum_{i=1}^{2} q_i T_i
\]

where \( q_1 = P(a_k=0) \) and \( q_2 = P(a_k=1) \).

- For the NRZI signal with equal state probabilities \( p_1 = p_2 = 0.5 \),

\[
P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
The transition probability matrix for the Miller-coded signal with equally likely symbols ($q_1=q_2=0.5$ or, equivalently, $p_1=p_2=p_3=p_4=0.25$) is:

$$
P = \begin{bmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0
\end{bmatrix}
$$

The transition probability matrix is useful in the determination of the spectral characteristics of digital modulation techniques with memory.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

Introduction

- In this section, we consider a class of digital modulation methods in which the phase of the signal is constrained to be continuous.
- This constraint results in a phase or frequency modulator that has memory.
- The modulation method is also non-linear.

Continuous-phase FSK (CPFSK)

- A conventional FSK signal is generated by shifting the carrier by an amount \( f_n = \frac{1}{2} \Delta f \cdot I_n \), \( I_n = \pm 1, \pm 3, \ldots, \pm (M - 1) \), to reflect the digital information that is being transmitted.
- This type (conventional type) of FSK signal is memoryless.
Continuous-phase FSK (CPFSK) (cont.)

- The switching from one frequency to another may be accomplished by having $M=2^k$ separate oscillators tuned to the desired frequencies and selecting one of the $M$ frequencies according to the particular $k$-bit symbol that is to be transmitted in a signal interval of duration $T=k/R$ seconds.

- The reasons why we have CPFSK: (or the defects of conventional FSK)
  - Such abrupt switching from one oscillator output to another in successive signaling intervals results in relatively large spectral side lobes outside of the main spectral band of the signal.
  - Consequently, this method requires a large frequency band for transmission of the signal.
Continuous-phase FSK (CPFSK) (cont.)

Solution:

To avoid the use of signals having large spectral side lobes, the information-bearing signal frequency modulates a single carrier whose frequency is changed continuously.

The resulting frequency-modulated signal is phase-continuous and, hence, it is called continuous-phase FSK (CPFSK).

This type (continuous-phase type) of FSK signal has memory because the phase of the carrier is constrained to be continuous.
Continuous-phase FSK (CPFSK) (cont.)

- In order to represent a CPFSK signal, we begin with a PAM signal:
  \[ d(t) = \sum_{n} I_n g(t - nT) \]
- \(d(t)\) is used to frequency-modulate the carrier.
- \(\{I_n\}\) denotes the sequence of amplitudes obtained by mapping \(k\)-bit blocks of binary digits from the information sequence \(\{a_n\}\) into the amplitude levels \(\pm 1, \pm 3, \ldots, \pm (M-1)\).
- \(g(t)\) is a rectangular pulse of amplitude \(1/2T\) and duration \(T\) seconds.
Continuous-phase FSK (CPFSK) (cont.)

- Equivalent low-pass waveform \( v(t) \) is expressed as

\[
v(t) = \sqrt{\frac{2\varepsilon}{T}} \exp\left\{ j\left[ 4\pi T f_d \int_{-\infty}^{t} d(\tau) d\tau + \phi_0 \right] \right\}
\]

- \( f_d \) is the peak frequency deviation, \( \phi_0 \) is the initial phase of the carrier.

- The carrier-modulated signal may be expressed as

\[
s(t) = \sqrt{\frac{2\varepsilon}{T}} \cos\left[ 2\pi f_c t + \phi(t; I) + \phi_0 \right]
\]

where \( \phi(t; I) \) represents the time-varying phase of the carrier.
Continuous-phase FSK (CPFSK) (cont.)

\[ \phi(t; I) = 4\pi f_d \int_{-\infty}^{t} d(\tau) d\tau = 4\pi f_d \int_{-\infty}^{t} \left[ \sum_{n} I_n g(\tau - nT) \right] d\tau \]

\[ = 2\pi f_d T \sum_{k=-\infty}^{n-1} I_k + 2\pi f_d (t - nT) I_n \quad \text{for} \quad nT \leq t \leq (n+1)T \]

\[ \theta_n = \pi h \sum_{k=-\infty}^{n-1} I_k \]

\[ h = 2 f_d T \]

Note that, although \( d(t) \) contains discontinuities, the integral of \( d(t) \) is continuous. Hence, we have a continuous-phase signal.

\[ \theta_n \] represents the accumulation (memory) of all symbols up to time \( nT \).

Parameter \( h \) is called the modulation index.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

Continuous-phase modulation (CPM)

- CPFSK becomes a special case of a general class of *continuous-phase modulated* (CPM) signals in which the carrier phase is

\[ \phi(t; I) = 2\pi \sum_{k=-\infty}^{n} I_k h_k q(t - kT), \quad nT \leq t \leq (n + 1)T \]

- when \( h_k = h \) for all \( k \), the modulation index is fixed for all symbols.
- when \( h_k \) varies from one symbol to another, the CPM signal is called *multi-\( h \)*. (In such a case, the \( \{h_k\} \) are made to vary in a cyclic manner through a set of indices.)

- The waveform \( q(t) \) may be represented in general as the integral of some pulse \( g(t) \), i.e.,

\[ q(t) = \int_{0}^{t} g(\tau) d\tau \]
Continuous-phase modulation (CPM) (cont.)

- If $g(t) = 0$ for $t > T$, the CPM signal is called full response CPM. (Fig a. b.)
- If $g(t) \neq 0$ for $t > T$, the modulated signal is called partial response CPM. (Fig c. d.)
Continuous-phase modulation (CPM) (cont.)

- The CPM signal has memory that is introduced through the phase continuity.
- For $L > 1$, additional memory is introduced in the CPM signal by the pulse $g(t)$.
- Three popular pulse shapes are given in the following table.
  - LREC denotes a rectangular pulse of duration $LT$.
  - LRC denotes a raised cosine pulse of duration $LT$.
  - Gaussian minimum-shift keying (GMSK) pulse with bandwidth parameter $B$, which represents the -3 dB bandwidth of the Gaussian pulse.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Continuous-phase modulation (CPM) (cont.)
  - Some commonly used CPM pulse shapes
  - **LREC**
    \[
g(t) = \begin{cases} 
    \frac{1}{2LT} & (0 \leq 1 \leq LT) \\
    0 & \text{(otherwise)} 
    \end{cases}
\]
  - **LRC**
    \[
g(t) = \begin{cases} 
    \frac{1}{2LT} \left(1 - \cos \frac{2\pi t}{LT}\right) & (0 \leq 1 \leq LT) \\
    0 & \text{(otherwise)} 
    \end{cases}
\]
  - **GMSK**
    \[
g(t) = \left\{ Q \left[ 2\pi B \left( t - \frac{T}{2} \right) \left( \ln 2 \right)^{1/2} \right] - Q \left[ 2\pi B \left( t + \frac{T}{2} \right) \left( \ln 2 \right)^{1/2} \right] \right\}
\]
    \[
    Q(t) = \int_{t}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx
    \]
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

Continuous-phase modulation (CPM) (cont.)
It is instructive to sketch the set of phase trajectories \( \phi (t;I) \)
generated by all possible values of the information sequence \( \{I_n\} \).

These phase diagrams are called \textit{phase tree}.

CPFSK with binary symbols \( I_n = \pm 1 \), the set of phase trajectories beginning at time \( t=0 \).

Phase trajectories for quaternary CPFSK.
We observe that the phase trees for CPFSK are piecewise linear as a consequence of the fact that the pulse \( g(t) \) is rectangular.

Smother phase trajectories and phase trees are obtained by using pulses that do not contain discontinuities, such as the class of raised cosine pulses.

For example, a phase trajectory generated by the sequence \((1, -1, -1, -1, 1, 1, -1, 1)\) for a partial response CPM based on raised cosine pulse of length \(3T\) is illustrated (solid).

Binary CPFSK (dashed)
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- The phase trees shown in these figures grow with time. However, the phase of the carrier is unique only in the range from $\phi = 0$ to $\phi = 2\pi$ (or from $\phi = -\pi$ to $\phi = \pi$).

- To properly view the phase trellis (the structure which the phase tree collapses into) diagram, we may plot the two quadrature components $x_c(t;I) = \cos \phi (t;I)$ and $x_s(t;I) = \sin \phi (t;I)$ as functions of time.

- Thus, we generate a 3-D plot in which the quadrature components $x_c$ and $x_s$ appear on the surface of a cylinder of unit radius.

The phase trellis or phase cylinder for binary CPM with $h=1/2$ and a raised cosine pulse of length $3T$. 
Simpler representations for the phase trajectories can be obtained by displaying only the terminal values of the signal phase at the time instants \( t=nT \).

In this case, we restrict the modulation index of the CPM signal to be rational. In particular, let us assume that \( h=m/p \), where \( m \) and \( p \) are relatively prime integers.

A full response CPM signal at the time instants \( t=nT \) will have the terminal phase states

\[
\Theta_s = \left\{ 0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \ldots, \frac{(p-1)\pi m}{p} \right\}, \quad \text{when } m \text{ is even}
\]

\[
\Theta_s = \left\{ 0, \frac{\pi m}{p}, \frac{2\pi m}{p}, \ldots, \frac{(2p-1)\pi m}{p} \right\}, \quad \text{when } m \text{ is odd}
\]
There are $p$ terminal phase states when $m$ is even and $2p$ states when $m$ is odd.

On the other hand, when the pulse shape extends over $L$ symbol intervals (partial response CPM), the number of phase states may increase up to a maximum of $S_t$, where

$$S_t = \begin{cases} 
pM^{L-1} & \text{(even } m) \\
2pM^{L-1} & \text{(odd } m) \end{cases}$$

where $M$ is the alphabet size.

For example, the binary CPFSK signal (full response, rectangular pulse) with $h=1/2$, has $S_t=4$ (terminal) phase states. The *state trellis* for this signal is illustrated.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- We emphasize that the phase transitions from one state to another are not true phase trajectories. They represent phase transitions for the (terminal) states at the time instants $t=nT$.

- An alternative representation to the state trellis is the state diagram. (also illustrates the state transitions at time instants $t=nT$)

- Characteristics of state diagram:
  - More compact.
  - Only possible (terminal) phase states and their transitions are displayed.
  - Time does not appear explicitly as a variable.

![State diagram for binary CPFSK with $h=1/2$]
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

**Minimum-shift keying (MSK).**

- MSK is a special form of binary CPFSK (and, therefore, CPM) in which the modulation index $h=1/2$.
- The phase of the carrier in the interval $nT \leq t \leq (n+1)T$ is

\[
\phi(t; I) = \frac{1}{2} \pi \sum_{k=-\infty}^{n-1} I_k + \pi I_n q(t-nT)
\]

\[
= \theta_n + \frac{1}{2} \pi I_n \left( \frac{t-nT}{T} \right), \quad nT \leq t \leq (n+1)T
\]

- The modulated carrier signal is

\[
s(t) = A \cos \left[ 2\pi f_c t + \theta_n + \frac{1}{2} \pi I_n \left( \frac{t-nT}{T} \right) \right]
\]

\[
= A \cos \left[ 2\pi \left( f_c + \frac{1}{4T} I_n \right) t - \frac{1}{2} n\pi I_n + \theta_n \right], \quad nT \leq t \leq (n+1)T
\]
Minimum-shift keying (MSK) (cont.)

The expression indicates that the binary CPFSK signal can be expressed as a sinusoid having one of two possible frequencies in the interval $nT \leq t \leq (n+1)T$. If we define these frequencies as

$$f_1 = f_c + \frac{1}{4T}$$

$$f_2 = f_c - \frac{1}{4T}$$

Then the binary CPFSK signal may be written in the form

$$s_i(t) = A \cos \left[ 2\pi f_i t + \theta_n + \frac{1}{2} n\pi (-1)^{i-1} \right], \quad i = 1, 2$$
Minimum-shift keying (MSK) (cont.)

- Why binary CPFSK with $h=1/2$ is called minimum-shift keying (MSK)?
- Because the frequency separation $\Delta f = f_2 - f_1 = 1/2T$, and $\Delta f = 1/2T$ is the minimum frequency separation that is necessary to ensure the orthogonality of the signals $s_1(t)$ and $s_2(t)$ over a signaling interval of length $T$.
- The phase in the $n$th signaling interval is the phase state of the signal that results in phase continuity between adjacent interval.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- **Minimum-shift keying (MSK) (Haykin)**
  - Consider a continuous-phase frequency-shift keying (CPFSK) signal, which is defined for the interval $0 \leq t \leq T_b$ as follows:

  $$s(t) = \begin{cases} 
  \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_1 t + \theta(0)] & \text{for symbol 1} \\
  \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_2 t + \theta(0)] & \text{for symbol 0}
  \end{cases}$$

  - $E_b$ is the transmitted signal energy per bit.
  - $T_b$ is the bit duration.
  - The phase $\theta(0)$, denoting the value of the phase at time $t = 0$, sums up the past history of the modulation process up to time $t = 0$. 
  

Another useful way of representing the CPFSK signal \( s(t) \) is to express it in the conventional form of an angle-modulated signal as follows:

\[
s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_c t + \theta(t)]
\]

\( \theta(t) \) is the phase of \( s(t) \).

When the phase \( \theta(t) \) is a continuous function of time, we find that the modulated signal \( s(t) \) itself is also continuous at all times, including the inter-bit switching times.
Minimum-shift keying (MSK) (Haykin)

- The phase $\theta(t)$ of a CPFSK signal increases or decreases linearly with time during each bit duration of $T_b$ seconds

$$\theta(t) = \theta(0) \pm \frac{\pi h}{T_b} t, \quad 0 \leq t \leq T_b$$

The plus (minus) sign corresponds to sending symbol 1 (0).

- We can find that

$$f_c + \frac{h}{2T_b} = f_1 \quad f_c - \frac{h}{2T_b} = f_2$$

- We thus get

$$f_c = \frac{1}{2}(f_1 + f_2) \quad h = T_b(f_1 + f_2)$$
Minimum-shift keying (MSK) (Haykin)

- $h$ is referred to as the deviation ratio.
- With $h=1/2$, the frequency deviation equals half the bit rate. This is the minimum frequency spacing that allows the two FSK signals representing symbols 1 and 0 to be coherently orthogonal.

At time $t = T_b$:

$$\theta(T_b) - \theta(0) = \begin{cases} 
\pi h & \text{for symbol 1} \\
-\pi h & \text{for symbol 0}
\end{cases}$$

- This is to say, the sending of symbol 1 increases the phase of a CPFSK signal $s(t)$ by $\pi h$ radians, whereas the sending of symbol 0 reduces it by an equal amount.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Minimum-shift keying (MSK) (Haykin)
- Phase Trellis
Minimum-shift keying (MSK) (Haykin)

Phase trellis of the sequence 1101000 ($h = 1/2$)

From (*), we may express the CPFSK signal $s(t)$:

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos(\theta(t)) \cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} \sin(\theta(t)) \sin(2\pi f_c t)$$

$$\equiv S_I \cos(2\pi f_c t) - S_Q \sin(2\pi f_c t) \quad (***)$$
Minimum-shift keying (MSK) (Haykin)

Consider first the in-phase component $S_I$

- With the deviation ratio $h=1/2$,
  \[
  \theta(t) = \theta(0) \pm \frac{\pi}{2T_b} t, \quad 0 \leq t \leq T_b
  \]

  where the plus (minus) sign corresponds to symbol 1 (0).

- A similar result holds for $\theta(t)$ in the interval $-T_b \leq t \leq 0$, except that the algebraic sign is not necessarily the same in both intervals.

- Since the phase $\theta(0)$ is 0 or $\pi$, depending on the past history of the modulation process, we find that, in the interval $-T_b \leq t \leq T_b$, the polarity of $\cos[\theta(t)]$ depends only on $\theta(0)$, regardless of the sequence of 1s and 0s transmitted before or after $t = 0$. 

4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM
Minimum-shift keying (MSK) (Haykin)

Thus, for $-T_b \leq t \leq T_b$, the in-phase component $s_I(t)$ consists of a half-cycle cosine pulse defined as follows:

$$s_I(t) = \pm \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi}{2T_b} t\right), \quad -T_b \leq t \leq T_b$$

where the plus (minus) sign corresponds to $\theta(0) = 0$ ($\theta(0) = \pi$).
Minimum-shift keying (MSK) (Haykin)

Similarly, for $0 \leq t \leq 2T_b$, the quadrature component $s_Q(t)$ consists of a half-cycle sine pulse, whose polarity depends only on $\theta(T_b)$, defined as follows:

$$s_Q(t) = \sqrt{\frac{2E_b}{T_b}} \sin \left[ \theta(t) \right] = \sqrt{\frac{2E_b}{T_b}} \sin \left[ \theta(T_b) \right] \sin \left( \frac{\pi}{2T_b} t \right)$$

$$= \pm \sqrt{\frac{2E_b}{T_b}} \sin \left( \frac{\pi}{2T_b} t \right), \quad 0 \leq t \leq 2T_b$$

where the plus (minus) sign corresponds to $\theta(T_b) = \pi/2$ ($\theta(T_b) = -\pi/2$).
Minimum-shift keying (MSK) (Haykin)

MSK signal may assume any one of four possible forms:

- The phase $\theta (0)=0$ and $\theta (T_b)=\pi /2$, corresponding to the transmission of symbol 1.
- The phase $\theta (0)=\pi$ and $\theta (T_b)=\pi /2$, corresponding to the transmission of symbol 0.
- The phase $\theta (0)=\pi$ and $\theta (T_b)=-\pi /2$, corresponding to the transmission of symbol 1.
- The phase $\theta (0)=0$ and $\theta (T_b)=-\pi /2$, corresponding to the transmission of symbol 0.
Minimum-shift keying (MSK) (Haykin)

Define the orthonormal basis functions:

\[
\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos \left( \frac{\pi}{2T_b} t \right) \cos(2\pi f_c t), \quad 0 \leq t \leq T_b \\
\phi_2(t) = \sqrt{\frac{2}{T_b}} \sin \left( \frac{\pi}{2T_b} t \right) \sin(2\pi f_c t), \quad 0 \leq t \leq T_b
\]

From (**), we have:

\[
s(t) = s_1 \phi_1(t) + s_2 \phi_2(t), \quad 0 \leq t \leq T_b
\]

where the coefficients \(s_1\) and \(s_2\) are related to the phase states \(\theta(0)\) and \(\theta(T_b)\), respectively.
Minimum-shift keying (MSK) (Haykin)

\[ s_1 = \int_{-T_b}^{T_b} s(t) \phi_1(t) \, dt = \sqrt{E_b} \cos[\theta(0)], \quad -T_b \leq t \leq T_b \]

\[ s_2 = \int_{0}^{2T_b} s(t) \phi_2(t) \, dt = -\sqrt{E_b} \sin[\theta(T_b)], \quad 0 \leq t \leq 2T_b \]

Both integrals are evaluated for a time interval equal to twice the bit duration.

Both the lower and upper limits of the product integration used to evaluate the coefficient \( s_1 \) are shifted by the bit duration \( T_b \) with respect to those used to evaluate the coefficient \( s_2 \).

The time interval \( 0 \leq t \leq T_b \), for which the phase states \( \theta(0) \) and \( \theta(T_b) \) are defined, is common to both integrals.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Minimum-shift keying (MSK) (Haykin)
- Signal space diagram for MSK system
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

Minimum-shift keying (MSK) (Haykin)

In QPSK the transmitted symbol is represented by any one of the four message points, whereas in MSK one of two message points is used to represent the transmitted symbol at any one time.

<table>
<thead>
<tr>
<th>Transmitted Binary Symbol, $0 \leq t \leq T_b$</th>
<th>Phase States (radians)</th>
<th>Coordinates of Message Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$\theta(0)$ $0$, $\theta(T_b)$ $-\pi/2$</td>
<td>$s_1$ $+\sqrt{E_b}$, $s_2$ $+\sqrt{E_b}$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\pi$, $-\pi/2$</td>
<td>$s_1$ $-\sqrt{E_b}$, $s_2$ $+\sqrt{E_b}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\pi$, $+\pi/2$</td>
<td>$s_1$ $-\sqrt{E_b}$, $s_2$ $-\sqrt{E_b}$</td>
</tr>
<tr>
<td>$1$</td>
<td>$0$, $+\pi/2$</td>
<td>$s_1$ $+\sqrt{E_b}$, $s_2$ $-\sqrt{E_b}$</td>
</tr>
</tbody>
</table>
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

Figure 6.30 (a) Input binary sequence. (b) Waveform of scaled time function $s_1 \phi_1(t)$. (c) Waveform of scaled time function $s_2 \phi_2(t)$. (d) Waveform of the MSK signal $s(t)$ obtained by adding $s_1 \phi_1(t)$ and $s_2 \phi_2(t)$ on a bit-by-bit basis.
Minimum-shift keying (MSK) (Haykin)

- Coherent detection of the MSK signal

- In the case of an AWGN channel, the received signal is given by

\[ x(t) = s(t) + w(t) \]

where \( s(t) \) is the transmitted MSK signal, and \( w(t) \) is the sample function of a white Gaussian noise process of zero mean and power spectral density \( N_0/2 \).

- For the optimal decision of \( \theta(0) \),

\[
 x_1 = \int_{-T_b}^{T_b} x(t) \phi_1(t) \, dt = s_1 + w_1, \quad -T_b \leq t \leq T_b
\]

If \( x_1 > 0 \) \( \Rightarrow \hat{\theta}(0) = 0 \)

If \( x_1 < 0 \) \( \Rightarrow \hat{\theta}(0) = \pi \)
Minimum-shift keying (MSK) (Haykin)

Coherent detection of the MSK signal (cont.)

For the optimum detection of $\theta (T_b)$,

$$x_2 = \int_0^{2T_b} x(t) \phi_2(t) \, dt = s_2 + w_2, \quad 0 \leq t \leq 2T_b$$

If $x_2 > 0 \implies \hat{\theta}(T_b) = -\pi / 2$

If $x_2 < 0 \implies \hat{\theta}(T_b) = \pi / 2$

Error probability of MSK

$$P_e = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} \right)$$

which is exactly the same as that for binary PSK and QPSK.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Minimum-shift keying (MSK) (Haykin)
- Block diagrams for MSK transmitter
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Minimum-shift keying (MSK) (Haykin)
- Block diagrams for coherent MSK receiver

![Block diagram for coherent MSK receiver]
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Minimum-shift keying (MSK) (Haykin)
- Power spectra of MSK signals

\[
S_B(f) = 2 \left( \frac{\psi_g(f)}{2T_b} \right) = \frac{32E_b}{\pi^2} \left[ \cos\left(2\pi T_b f\right) \right]^2
\]

MSK does not produce as much interference outside the signal band of interest as QPSK.
Minimum-shift keying (MSK) (cont.)

MSK may also be represented as a form of four-phase PSK.

The equivalent low-pass digitally modulated signal in the form

\[ v(t) = \sum_{n=-\infty}^{\infty} \left[ I_{2n} g(t - 2nT) - jI_{2n+1} g(t - 2nT - T) \right] \]

\[ g(t) \] is a sinusoidal pulse

\[ g(t) = \begin{cases} 
\sin \frac{\pi t}{2T} & (0 \leq t \leq 2T) \\
0 & \text{(otherwise)}
\end{cases} \]
Minimum-shift keying (MSK) (cont.)

- This type of signal is viewed as a four-phase PSK signal in which the pulse shape is one-half cycle of a sinusoid ($0 \sim \pi$).
- The even-numbered binary-valued ($\pm 1$) symbols $\{I_{2n}\}$ of the information sequence $\{I_n\}$ are transmitted via the cosine of the carrier, while the odd-numbered symbols $\{I_{2n+1}\}$ are transmitted via the sine of the carrier.
- The transmission rate on the two orthogonal carrier components is $1/2T$ bits/s so that the combined transmission rate is $1/T$ bits/s.
- Note that the bit transitions on the sine and cosine carrier components are staggered or offset in time by $T$ seconds.
Minimum-shift keying (MSK) (cont.)

For this reason, the signal

\[ s(t) = A \left\{ \sum_{n=-\infty}^{\infty} I_{2n} g(t - 2nT) \right\} \cos 2\pi f_c t + \left\{ \sum_{n=-\infty}^{\infty} I_{2n+1} g(t - 2nT - T) \right\} \sin 2\pi f_c t \]

is called offset quadrature PSK (OQPSK) or staggered quadrature PSK (SQPSK).

Figure in next page illustrates the representation of an MSK signal as two staggered quadrature-modulated binary PSK signals. The corresponding sum of the two quadrature signals is a constant amplitude, frequency-modulated signal.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Minimum-shift keying (MSK) (cont.)

(a) In-phase signal component

(b) Quadrature signal component

(c) MSK signal [sum of (a) and (b)]
Minimum-shift keying (MSK) (cont.)

- Compare the waveforms for MSK with OQPSK (pulse $g(t)$ is rectangular for $0 \leq t \leq 2T$) and with conventional QPSK (pulse $g(t)$ is rectangular for $0 \leq t \leq 2T$).
  - All three of the modulation methods result in identical data rates.
  - The MSK signal has continuous phase.
  - The OQPSK signal with a rectangular pulse is basically two binary PSK signals for which the phase transitions are staggered in time by $T$ seconds. Thus, the signal contains phase jumps of $\pm 90^\circ$.
  - The conventional four-phase PSK (QPSK) signal with constant amplitude will contain phase jumps of $\pm 180^\circ$ or $\pm 90^\circ$ every $2T$ seconds.
4.3.3 Non-linear Modulation Methods with Memory—CPFSK and CPM

- Minimum-shift keying (MSK) (cont.)
  - Compare the waveforms for MSK with OQPSK and QPSK (cont.)

---

(a) MSK

(b) Offset QPSK

(c) QPSK

-90° phase shift
+90° phase shift

-90° phase shift
+90° phase shift

180° phase shift
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

Minimum-shift keying (MSK) (cont.)

- QPSK/OQPSK
- MSK/FFSK

Graph showing frequency spectrum with labels:
- $f_c - \frac{3}{4T_b}$
- $f_c + \frac{3}{4T_b}$
- $f_c - \frac{1}{2T_b}$
- $f_c + \frac{1}{2T_b}$

Signal space diagrams for CPM.

- In general, continuous-phase signals cannot be represented by discrete points in signal space as in the case of PAM, PSK, and QAM, because the phase of the carrier is time-variant.
- Instead, a continuous-phase signal is described by the various paths or trajectories from one phase state to another.
- For a constant-amplitude CPM signal, the various trajectories form a circle.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Signal space diagrams for CPM (cont.)
  - For example, figures in the right illustrate the signal space (phase trajectory) diagram for CPFSK signals.
  - Note that the length of the phase trajectory increases with an increase in $h$. An increase in $h$ also results in an increase of the signal bandwidth, as demonstrated in the following section.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- A linear representation of CPM.
  - CPM is a non-linear modulation technique with memory.
  - CPM may also be represented as a linear superposition of signal waveforms.
  - Such a representation provides an alternative method for generating the modulated signal at the transmitter and/or demodulating the signal at the receiver.
  - We demonstrate that binary CPM may be represented by a linear superposition of a finite number of amplitude-modulated pulses, provided that the pulse $g(t)$ is of finite duration $LT$, where $T$ is the bit interval.
A linear representation of CPM (cont.)

We begin with the equivalent low-pass representation of CPM

\[ v(t) = \sqrt{\frac{2\varepsilon}{T}} e^{j\phi(t; I)}, \quad nT \leq t \leq (n+1)T \]

where

\[ \phi(t; I) = 2\pi h \sum_{k=-\infty}^{n} I_k q(t-kT) , \quad nT \leq t \leq (n+1)T \]

\[ = \pi h \sum_{k=-\infty}^{n-L} I_k + 2\pi h \sum_{k=n-L+1}^{n} I_k q(t-kT) \]

and \( q(t) \) is the integral of the pulse \( g(t) \).
A linear representation of CPM (cont.)

The exponential term may be expressed as

\[
\exp\left[j\phi(t; I)\right] = \exp\left(j\pi h \sum_{k=-\infty}^{n-L} I_k\right) \prod_{k=0}^{L-1} \exp\left\{ j2\pi h L_{n-k}q [t - (n - k)T] \right\}
\]

Note that the first term on the right-hand side represents the cumulative phase up to the information symbol \(I_{n-L}\), and the second term consists of a product of \(L\) phase terms.
A linear representation of CPM (cont.)

Assuming that the modulation index $h$ is not an integer and the data symbol are binary, i.e., $I_k = \pm 1$, the $k$th phase term may be expressed as

$$\exp \left\{ j 2\pi h I_{n-k} q \left[ t - (n-k)T \right] \right\}$$

$$= \frac{\sin \pi h}{\sin \pi h} \exp \left\{ j 2\pi h I_{n-k} q \left[ t - (n-k)T \right] \right\}$$

$$= \frac{\sin \{ \pi h - 2\pi h q [t - (n-k)] T \}}{\sin \pi h}$$

$$+ \exp(j \pi h I_{n-k}) \frac{\sin \{ 2\pi h q [t - (n-k)T] \}}{\sin \pi h}$$
A linear representation of CPM (cont.)

It is convenient to define the signal pulse $s_0(t)$ as

$$s_0(t) = \begin{cases} 
\sin 2\pi h q(t) & (0 \leq t \leq LT) \\
\sin \pi h \sin[\pi h - 2\pi h q(t - LT)] & (LT \leq t \leq 2LT) \\
0 & \text{(otherwise)}
\end{cases}$$

Then,

$$\exp[j\phi(t; I)] = \exp \left( j\pi h \sum_{k=0}^{n-L} I_k \right) \prod_{k=0}^{L-1} \left\{ s_0 \left[ t + (k + L - n)T \right] + \exp(j\pi h I_{n-k}) s_0 \left[ t - (k - n)T \right] \right\}$$
A linear representation of CPM (cont.)

Then, we obtain a sum of $2^L$ terms, where $2^{L-1}$ terms are distinct and the other $2^{L-1}$ terms are time-shifted versions of the distinct terms.

The final result may be expressed as

$$\exp[j\phi(t; I)] = \sum_{n} \sum_{k=0}^{2^{L-1}-1} e^{j\pi h A_{k,n}} c_k(t - nT)$$

For $0 \leq k \leq 2^{L-1}-1$, the pulses $c_k(t)$ are defined as

$$c_k(t) = s_0(t) \prod_{n=1}^{L-1} s_0 \left[ t + (n + La_{k,n}) T \right], \quad 0 \leq t \leq T \cdot \min_{\{n\}} \left[ L(2 - a_{k,n}) - n \right]$$
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- A linear representation of CPM (cont.)
  - Each pulse is weighted by a complex coefficient $\exp(j \pi hA_{k,n})$, where
    \[ A_{k,n} = \sum_{m=-\infty}^{n} I_m - \sum_{m=1}^{L-1} I_{n-m} a_{k,m} \]
  - And the $\{a_{k,n} = 0 \text{ or } 1\}$ are the coefficients in the binary representation of the index $k$,
    \[ k = \sum_{m=1}^{L-1} 2^{m-1} a_{k,m}, \quad k = 0,1, \ldots, 2^{L-1} - 1 \]
  - Thus, the binary CPM signal is expressed as a weighted sum of $2^{L-1}$ real-valued pulses $\{c_k(t)\}$. 
A linear representation of CPM (cont.)

- In this representation, the pulse $c_0(t)$ is the most important component, because its duration is the longest and it contains the most significant part of the signal energy.
- Consequently, a simply approximation to a CPM signal is a partial response PAM signal having $c_0(t)$ as the basic pulse shape.
- The focus for the above development was binary PCM.

**Example 4.3-1** As a special case, let us consider the MSK signal, for which $h=1/2$ and $g(t)$ is a rectangular pulse of duration $T$. In this case,

$$\phi(t; I) = \frac{\pi}{2} \sum_{k=-\infty}^{n-1} I_k + \pi I_n q(t - nT)$$

$$= \theta_n + \frac{\pi}{2} I_n \left( \frac{t - nT}{T} \right), \quad nT \leq t \leq (n + 1)T$$
Example 4.3-1 (cont.)

And \[ \exp[j \phi(t; I)] = \sum_{n} b_n c_0(t - nT) \]

where \( c_0(t) = \begin{cases} \sin \frac{\pi t}{2T} & (0 \leq t \leq 2T) \\ 0 & \text{(otherwise)} \end{cases} \)

\[ b_n = e^{j \pi A_{0,n}/2} = e^{j \pi (\theta_n + I_n)/2} \]

The complex-valued modified data sequence \( \{b_n\} \) may be expressed recursively as

\[ b_n = j b_{n-1} I_n \]

so that \( b_n \) alternates in taking real and imaginary values.
Multiamplitude CPM.

Multiamplitude CPM is a generalization of ordinary CPM in which the signal amplitude is allowed to vary over a set of amplitude values while the phase of the signal is constrained to be continuous.

For example, consider a two-amplitude CPFSK signal,

\[ s(t) = 2A \cos[2\pi f_c t + \phi_2(t; I)] + A \cos[2\pi f_c t + \phi_1(t; J)] \]

where

\[ \phi_2(t; I) = \pi h \sum_{k=-\infty}^{n-1} I_k + \frac{\pi h I_n (t - nT)}{T}, \quad nT \leq t \leq (n+1)T \]

\[ \phi_1(t; J) = \pi h \sum_{k=-\infty}^{n-1} J_k + \frac{\pi h J_n (t - nT)}{T}, \quad nT \leq t \leq (n+1)T \]
Multiamplitude CPM (cont.)

- The information is conveyed by the symbol sequences \{I_n\} and \{J_n\}, which are related to two independent binary information sequences \{a_n\} and \{b_n\}, that take values \{0, 1\}.
- The sequences \{I_n\} and \{J_n\} are not statistically independent, but are constrained in order to achieve phase continuity in the superposition of the two components.
- Consider the case where \( h = 1/2 \) so that we have the superposition of two MSK signals.
- The phase change in the signal is determined by the phase of the larger amplitude component, while the amplitude change is determined by the smaller component.
**Multiamplitude CPM (cont.)**

- The smaller component is constrained such that at the start and end of each symbol interval, it is either in phase or $180^\circ$ out of phase with the larger component, independent of its phase.

- Under this constraint,
  
  \[ I_n = 2a_n - 1 \]

  \[ J_n = I_n (1 - 2b_n) = I_n \left(1 - \frac{b_n}{h}\right) \]

<table>
<thead>
<tr>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$I_n$</th>
<th>$J_n$</th>
<th>Amplitude–phase relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>−1</td>
<td>−1</td>
<td>Amplitude is constant; phase decreases</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>−1</td>
<td>1</td>
<td>Amplitude changes; phase decreases</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>Amplitude is constant; phase increases</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>−1</td>
<td>Amplitude changes; phase increases</td>
</tr>
</tbody>
</table>
Multiamplitude CPM (cont.)

A multiamplitude CPFSK signal with \( n \) components

\[
s(t) = 2^{N-1} \cos[2\pi f_c t + \phi_N(t; I)] + \sum_{m=1}^{N-1} 2^{m-1} \cos\left[2\pi f_c t + \phi_m(t; J_m)\right]
\]

where

\[
\phi_N(t; I) = \pi h I_n \frac{t - nT}{T} + \pi h \sum_{k=-\infty}^{n-1} I_k , \quad nT \leq t \leq (n+1)T
\]

\[
\phi_m(t; J_m) = I_n \pi \left[ h + \frac{1}{2} (J_{mn} + 1) \right] \frac{t - nT}{T}
+ \sum_{k=-\infty}^{n-1} \pi I_k \left[ h + \frac{1}{2} (J_{mk} + 1) \right] , \quad nT \leq t \leq (n+1)T
\]

The sequences \( \{I_n\} \) and \( \{J_{mn}\} \) are statistically independent, binary-valued sequences that take values from the set \( \{1, -1\} \).
Multiamplitude CPM (cont.)

- From equations, we observe that each component in the sum will be either in phase or $180^\circ$ out of phase with the largest component at the end of the $n$th symbol interval, i.e., at $t-(n+1)T$.

- The signal states are specified by an amplitude level from the set of amplitudes $\{1, 3, 5, \cdots, 2^{N-1}\}$ and a phase level from the set $\{0, \pi, 2\pi \theta, 2\pi, 2\pi - \pi h\}$.

- The phase constraint is required to maintain the phase continuity of the CPM signal.

- Additional multiamplitude CPM signal formats may be obtained by using pulse shapes other than rectangular, as well as signal pulses that span more than one symbol (partial response).
Multiamplitude CPM (cont.)

- Figures illustrate the signal space diagrams for two-amplitude ($N=2$) CPFSK with $h = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \text{ and } \frac{2}{3}$.
Multiamplitude CPM (cont.)

- Figures illustrate the signal space diagrams for three-amplitude ($N=3$) CPFSK with $h = \frac{1}{4}, \frac{1}{3}, \frac{1}{2},$ and $\frac{2}{3}$.
- In this case, there are four amplitude levels.
- The number of states depends on the modulation index $h$ as well as $N$. 
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Gaussian-filtered MSK (Haykin)
  - Desirable properties of the MSK signals
    - Constant envelope
    - Relatively narrow bandwidth
    - Coherent detection performance equivalent to that of QPSK
  - However, the out-of-band spectral characteristics of MSK signals still do not satisfy the stringent requirements of certain applications such as wireless communications.
  - Solution: modifying the power spectrum into a compact form using a premodulation low-pass filter (referred to as a baseband pulse-shaping filter).
Gaussian-filtered MSK (Haykin)

- The pulse-shaping filter should satisfy the following properties:
  - Frequency response with narrow bandwidth and sharp cutoff characteristics.
  - Impulse response with relatively low overshoot.
  - Evolution of a phase trellis where the carrier phase of the modulated signal assumes the two values $\pm \pi/2$ at odd multiples of $T_b$ and the two values 0 and $\pi$ at even multiples of $T_b$ as in MSK.

- The desirable properties can be achieved by passing a nonreturn-to-zero (NRZ) binary data stream through a baseband pulse-shaping filter whose impulse response is defined by a Gaussian function.
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

Gaussian-filtered MSK (Haykin)

- The resulting method of binary frequency modulation is referred as Gaussian-filtered MSK or just GMSK.
- The response of this Gaussian filter to a rectangular pulse of unit amplitude and duration $T_b$ is given by

$$g(t) = \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} h(t-\tau) \, d\tau = \sqrt{\frac{2\pi}{\log 2}} W \int_{-\frac{T_b}{2}}^{\frac{T_b}{2}} \exp\left(-\frac{2\pi^2}{\log 2} W^2 (t-\tau)^2\right) \, d\tau$$

- This function may be expressed as the difference between two complementary error functions:

$$g(t) = \frac{1}{2} \left[ \text{erfc}\left(\pi \sqrt{\frac{2}{\log 2}} W T_b \left(\frac{t}{T_b} - \frac{1}{2}\right)\right) - \text{erfc}\left(\pi \sqrt{\frac{2}{\log 2}} W T_b \left(\frac{t}{T_b} + \frac{1}{2}\right)\right) \right]$$
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

- Gaussian-filtered MSK (Haykin)

**Figure 6.32** Frequency-shaping pulse $g(t)$ of Equation (6.135) shifted in time by $2.5T_b$ and truncated at $\pm 2.5T_b$ for varying time-bandwidth product $WT_b$. 
4.3.3 Non-linear Modulation Methods with Memory—CPFSK and CPM

- Gaussian-filtered MSK (Haykin)

![Graph showing power spectral density for MSK and GMSK signals](image)

**Figure 6.33** Power spectra of MSK and GMSK signals for varying time-bandwidth product. (Reproduced with permission from Dr. Gordon Stüber, Georgia Tech.)
4.3.3 Non-linear Modulation Methods with Memory---CPFSK and CPM

Gaussian-filtered MSK (Haykin)

![Diagram of power spectrum of GMSK signal for GSM wireless communications.](image)
4.4 Spectral Characteristics of Digitally Modulated Signals

- In most digital communication systems, the available channel bandwidth is limited.

- The system designer must consider the constraints imposed by the channel bandwidth limitation in the selection of the modulation technique used to transmit the information.

- From the power density spectrum, we can determine the channel bandwidth required to transmit the information-bearing signal.
4.4.1 Power Spectra of Linearly Modulation Signals

- Beginning with the form
  \[ s(t) = \text{Re} \left[ \nu(t) e^{j2\pi f_c t} \right] \]
- where \( \nu(t) \) is the equivalent low-pass signal.
- Autocorrelation function (from 4.1-50)
  \[ \phi_{ss}(\tau) = \text{Re} \left[ \phi_{\nu\nu}(\tau) e^{j2\pi f_c \tau} \right] \]
- Power density spectrum (from 4.1-51)
  \[ \Phi_{ss}(f) = \frac{1}{2} \left[ \Phi_{\nu\nu}(f - f_c) + \Phi_{\nu\nu}(-f - f_c) \right] \]
- First we consider the general form
  \[ \nu(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT) \]
- where the transmission rate is \( 1/T = R/k \) symbols/s and \( \{I_n\} \) represents the sequence of symbols.
4.4.1 Power Spectra of Linearly Modulation Signals

Autocorrelation function

$$\phi_{\nu\nu}(t + \tau; t) = \frac{1}{2} E[\nu^*(t)\nu(t + \tau)]$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[I_n^*I_m]g^*(t - nT)g(t + \tau - mT)$$

We assume the \(\{I_n\}\) is WSS with mean \(\mu_i\) and the autocorrelation function

$$\phi_{ii}(m) = \frac{1}{2} E[I_n^*I_{n+m}]$$

$$\phi_{\nu\nu}(t + \tau; t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \phi_{ii}(m - n)g^*(t - nT)g(t + \tau - mT) \quad \text{let } m' = m - n$$

$$= \sum_{m'=-\infty}^{\infty} \phi_{ii}(m') \sum_{n=-\infty}^{\infty} g^*(t - nT)g(t + \tau - (m' + n)T) \quad \text{let } m = m'$$

$$= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} g^*(t - nT)g(t + \tau - nT - mT)$$
The second summation
\[ \sum_{n=-\infty}^{\infty} g^*(t - nT)g(t + \tau - nT - mT) \]
is periodic in the \( t \) variable with period \( T \).

Consequently, \( \phi_{\nu \nu}(t + \tau; t) \) is also periodic in the \( t \) variable with period \( T \). That is
\[ \phi_{\nu \nu}(t + T + \tau; t + T) = \phi_{\nu \nu}(t + \tau; t) \]

In addition, the mean value of \( \nu(t) \), which is
\[ E[\nu(t)] = E\left[ \sum_{n=-\infty}^{\infty} I_n g(t - nT) \right] = \mu_i \sum_{n=-\infty}^{\infty} g(t - nT) \]
is periodic with period \( T \).
Therefore \( \nu(t) \) is a stochastic process having a periodic mean and autocorrelation function. Such a process is called a \textit{cyclostationary process} or a \textit{periodically stationary process in the wide sense}.

In order to compute the power density spectrum of a cyclostationary process, the dependence of \( \phi_{\nu \nu}(t+\tau;t) \) on the \( t \) variable must be eliminated. Thus,

\[
\overline{\phi_{\nu \nu}(\tau)} = \frac{1}{T} \int_{-T/2}^{T/2} \phi_{\nu \nu}(t+\tau;t)dt
\]

\[
= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^*(t-nT)g(t+\tau-nT-mT)dt
\]

\[
= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-T/2-nT}^{T/2-nT} g^*(t')g(t'+\tau-mT)dt' \quad (t' = t-nT)
\]
4.4.1 Power Spectra of Linearly Modulation Signals

- We interpret the integral as the time-autocorrelation function of $g(t)$ and define it as
  \[
  \phi_{gg}(\tau) = \int_{-\infty}^{\infty} g^*(t)g(t+\tau)dt
  \]

- Consequently,
  \[
  \bar{\phi}_{\nu\nu}(\tau) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \phi_{ii}(m)\phi_{gg}(\tau-mT)
  \]

- The (average) power density spectrum of $\nu(t)$ is in the form
  \[
  \Phi_{\nu\nu}(f) = \frac{1}{T} |G(f)|^2 \Phi_{ii}(f)
  \]
  Applying equation 2.2-27

where $G(f)$ is the Fourier transform of $g(t)$, and $\Phi_{ii}(f)$ denotes the power density spectrum of the information sequence
\[
\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m)e^{-j2\pi fmT}
\]
The result illustrates the dependence of the power density spectrum of \( v(t) \) on the spectral characteristics of the pulse \( g(t) \) and the information sequence \( \{I_n\} \).

That is, the spectral characteristics of \( v(t) \) can be controlled by (1) design of the pulse shape \( g(t) \) and by (2) design of the correlation characteristics of the information sequence.

Whereas the dependence of \( \Phi_{vv}(f) \) on \( G(f) \) is easily understood upon observation of equation, the effect of the correlation properties of the information sequence is more subtle.

First of all, we note that for an arbitrary autocorrelation \( \phi_{ii}(m) \) the corresponding power density spectrum \( \Phi_{ii}(f) \) is periodic in frequency with period \( 1/T \). (see next page)
In fact, the expression relating the spectrum $\Phi_{ii}(f)$ to the autocorrelation $\phi_{ii}(m)$ is in the form of an exponential Fourier series with the $\{\phi_{ii}(m)\}$ as the Fourier coefficients.

$$\phi_{ii}(m) = T \int_{-T/2}^{T/2} \Phi_{ii}(f)e^{j2\pi fm} df$$

Second, let us consider the case in which the information symbols in the sequence are real and mutually uncorrelated. In this case, the autocorrelation function $\phi_{ii}(m)$ can be expressed as (applying 2.2-5 and 2.2-6)

$$\phi_{ii}(m) = \begin{cases} \sigma_i^2 + \mu_i^2 & (m = 0) \\ \mu_i^2 & (m \neq 0) \end{cases}$$

where $\sigma_i^2$ denotes the variance of an information symbol.
4.4.1 Power Spectra of Linearly Modulation Signals

Substitute for $\phi_{ii}(m)$ in equation, we obtain

$$\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m)e^{-j2\pi fmT}$$

where

$$\sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{j\omega_n t} = \sigma_i^2 + \mu_i^2 \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT}$$

$$= \sigma_i^2 + \mu_i^2 \sum_{m=-\infty}^{\infty} \delta(f - \frac{m}{T})$$

It may be viewed as the exponential Fourier series of a periodic train of impulses with each impulse having an area $1/T$.

The desired result for the power density spectrum of $\nu(t)$ when the sequence of information symbols is uncorrelated.

$$\Phi_{\nu
u}(f) = \frac{\sigma_i^2}{T} |G(f)|^2 + \frac{\mu_i^2}{T^2} \sum_{m=-\infty}^{\infty} \left| G\left(\frac{m}{T}\right) \right|^2 \delta(f - \frac{m}{T})$$

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The expression for the power density spectrum is purposely separated into two terms to emphasize the two different types of spectral components.

The first term is the continuous spectrum, and its shape depends only on the spectral characteristic of the signal pulse \( g(t) \).

The second term consists of discrete frequency components spaced \( 1/T \) apart in frequency. Each spectral line has a power that is proportional to \( |G(f)|^2 \) evaluated at \( f = m/T \).

Note that the discrete frequency components vanish when the information symbols have zero mean, i.e., \( \mu_i = 0 \). This condition is usually desirable for the digital modulation techniques under consideration, and it is satisfied when the information symbols are equally likely and symmetrically positioned in the complex plane.
Example 4.4-1  To illustrate the spectral shaping resulting from $g(t)$, consider the rectangular pulse shown in figure. The Fourier transform of $g(t)$ is

$$G(f) = AT \frac{\sin \pi f T}{\pi f T} e^{-j \pi f T}$$

Hence

$$|G(f)|^2 = (AT)^2 \left( \frac{\sin \pi f T}{\pi f T} \right)^2$$

Thus

$$\Phi_{uv}(f) = \sigma_i^2 A^2 T \left( \frac{\sin \pi f T}{\pi f T} \right)^2 + A^2 \mu_i^2 \delta(f)$$

Decays inversely as the square of the frequency.
Example 4.4-2  As a second illustration of the spectral shaping resulting from \( g(t) \), we consider the raised cosine pulse

\[
g(t) = \frac{A}{2} \left[ 1 + \cos \frac{2\pi}{T} \left( t - \frac{T}{2} \right) \right], \quad 0 \leq t \leq T
\]

its Fourier transform is:

\[
G(f) = \frac{A}{2} \frac{\sin \pi f T}{\pi f T (1 - f^2 T^2)} e^{-j\pi f T}
\]

Decays inversely as the \( f^6 \)

nonzero
Example 4.4-3  To illustrate that spectral shaping can also be accomplished by operations performed on the input information sequence, we consider a binary sequence \( \{b_n\} \) from which we form the symbols \( I_n = b_n + b_{n-1} \).

The \( \{b_n\} \) are assumed to be uncorrelated random variables, each having zero mean and unit variance. Then the autocorrelation function of the sequence \( \{I_n\} \) is

\[
\phi_{ii}(m) = E(I_n I_{n+m}) = E\left[(b_n + b_{n-1})(b_{n+m} + b_{n+m-1})\right]
\]

\[
= \begin{cases} 
E\left[b_n^2 + 2b_nb_{n-1} + b_{n-1}^2\right] & (m = 0) \\
E\left[b_n^2 + b_nb_{n+1} + b_{n-1}b_{n+1} + b_nb_{n-1}\right] & (m = +1) \\
E\left[bb_{n-1} + b_nb_{n-2} + b_{n-1}^2 + b_{n-1}b_{n-2}\right] & (m = -1) \\
E\left[bb_{n+m} + b_nb_{n+m-1} + b_{n-1}b_{n+m} + b_{n-1}b_{n+m-1}\right] & (\text{otherwise}) 
\end{cases}
\]

\[E[(X_i-0)^2]=1\]
Example 4.4-3 (cont.)

Hence, the power density spectrum of the input sequence is (from 4.4-13)

\[ \Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_i(m)e^{-j2\pi fmT} \]
\[ = 2(1 + \cos 2\pi fT) \]
\[ = 4\cos^2 \pi fT \]

and the corresponding power density spectrum for the (low-pass) modulated signal is (from 4.4-12)

\[ \Phi_{\nu\nu}(f) = \frac{4}{T}|G(f)|^2 \cos^2 \pi fT \]
In this section, we derive the power density spectrum for the class of constant amplitude CPM signals.

We begin by computing the autocorrelation function and its Fourier transform, as was done in the case of linearly modulated signals.

The constant amplitude CPM signal is expressed as

\[ s(t; I) = A \cos[2\pi f_c t + \phi(t; I)] \]

where

\[ \phi(t; I) = 2\pi h \sum_{k=-\infty}^{\infty} I_k q(t - kT) \]

Each symbol in the sequence \( \{I_n\} \) can take one of the \( M \) values \( \{\pm 1, \pm 3, \cdots, \pm(M-1)\} \).
4.4.2 Power Spectra of CPFSK and CPM Signals

These symbols are statistically independent and identically distributed with prior probabilities

\[ P_n = P(I_k = n), \quad n = \pm 1, \pm 3, \ldots \ldots \pm (M - 1) \]

where \( \sum_n P_n = 1 \). The pulse \( g(t) = q'(t) \) is zero outside of the interval \([0, LT]\), \( q(t) = 0, \ t<0 \), and \( q(t)=1/2 \) for \( t>LT \).

The autocorrelation function of the equivalent low-pass signal

\[ \nu(t) = e^{j\phi(t;1)} \]

is

\[ \phi_{\nu\nu}(t + \tau, t) = \frac{1}{2} E \left[ \nu_{t+\tau} \nu_t^* \right] \]

\[ = \frac{1}{2} E \left[ \exp \left( j2\pi h \sum_{k=-\infty}^{\infty} I_k \left[ q(t+\tau-kT) - q(t-kT) \right] \right) \right] \]
First we express the sum in the exponent as a product of exponents.

\[
\phi_{vv}(t + \tau, t) = \frac{1}{2} E \left\{ \prod_{k=\infty}^{\infty} \exp\{ j2\pi h I_k [q(t + \tau - kT) - q(t - kT)]\} \right\}
\]

Next, we perform the expectation over the data symbols \( \{I_k\} \). Since these symbols are statistically independent, we obtain

\[
\phi_{vv}(t + \tau, t) = \frac{1}{2} \prod_{k=\infty}^{\infty} \left( \sum_{n=-(M-1)}^{M-1} P_n \exp\{ j2\pi h n [q(t + \tau - kT) - q(t - kT)]\} \right)
\]

Finally, the average autocorrelation function is

\[
\bar{\phi}_{vv}(\tau) = \frac{1}{T} \int_0^T \phi_{vv}(t + \tau; t) dt
\]
Although equation implies that there is an infinite number of factors in the product, the \( g(t) = q'(t) = 0 \) for \( t < 0 \) and \( t > LT \), \( q(t) = 0 \) for \( t < 0 \), and \( q(t) = \frac{1}{2} \) for \( t > LT \). Consequently only a finite number of terms in the product have nonzero exponents.

If we let \( \tau = \xi + mT \), where \( 0 \leq \xi < T \) and \( m=0,1,\cdots \), the average autocorrelation in equation reduces to

\[
\bar{\phi}_{vv}(\xi + mT) = \frac{1}{2T} \int_0^T \prod_{k=1-L}^{m+1} \sum_{n=-\left(M-1\right)}^{\left(M-1\right)} P_n \exp\left\{ j2\pi nhn [(q(t + \xi - (k - m)T) - q(t - kT)] \right\} dt
\]

\[ k \leq m+1 \quad \text{since} \quad \begin{cases} q_1 = q(t + \xi - (k - m)T) = 0 \quad \text{for} \quad 2 - k + m \leq 0 \quad (\because \quad 0 \leq t + \xi < 2T) \\ q_2 = q(t - kT) = 0 \quad \text{for} \quad 1 - k \leq 0 \end{cases} \]

\[ k \geq 1 - L \quad \text{since} \quad \begin{cases} q_1 = q(t + \xi - (k - m)T) = \frac{1}{2} \quad \text{for} \quad 0 - k + m \geq L \quad (\because \quad 0 \leq t + \xi < 2T) \\ q_2 = q(t - kT) = 1/2 \quad \text{for} \quad 0 - k \geq L \end{cases} \]
4.4.2 Power Spectra of CPFSK and CPM Signals

Let us focus on $\phi_{uv}(\xi + mT)$ for $\xi + mT \geq LT$. In this case, equation may be expressed as

$$\tilde{\phi}_{uv}(\xi + mT) = \left[\psi(jh)\right]^{m-L} \lambda(\xi), \quad m \geq L, \quad 0 \leq \xi \leq T$$

where $\psi(jh)$ is the characteristic function of the random sequence $\{I_n\}$,

$$\psi(jh) = E(e^{j\pi h I_n}) = \sum_{n=-(M-1)}^{M-1} P_n e^{j\pi h n}$$

$\lambda(\xi)$ is the remaining part of the average autocorrelation function

$$\lambda(\xi) = \frac{1}{2T} \int_0^T \left[ \prod_{k=1-L}^0 \sum_{n=- (M-1)} \right] P_n \exp\left\{ j2\pi h n \left[ \frac{1}{2} - q(t-kT) \right] \right\} \prod_{k=m+1-L}^{m+1} \left[ \sum_{n=- (M-1)} \right] P_n \exp\left[ j2\pi h n q(t+\xi - (k-m)T) \right] dt, \quad m \geq L$$
The Fourier transform of \( \bar{\phi}_{\nu\nu}(\tau) \) yields the average power density spectrum as

\[
\Phi_{\nu\nu}(f) = \int_{-\infty}^{\infty} \bar{\phi}_{\nu\nu}(\tau)e^{-j2\pi f \tau} d\tau
\]

\[
= 2 \Re \left[ \int_{0}^{\infty} \bar{\phi}_{\nu\nu}(\tau)e^{-j2\pi f \tau} d\tau \right]
\]

\( \Phi_{\nu\nu}(f) \) is real (2.2-19) and symmetric about \( f = 0 \) (4.1-54).

But

\[
\int_{0}^{\infty} \bar{\phi}_{\nu\nu}(\tau)e^{-j2\pi f \tau} d\tau = \int_{0}^{LT} \bar{\phi}_{\nu\nu}(\tau)e^{-j2\pi f \tau} d\tau
\]

\[
+ \int_{LT}^{\infty} \bar{\phi}_{\nu\nu}(\tau)e^{-j2\pi f \tau} d\tau
\]
The integral in the range \( LT \leq \tau \leq \infty \) may be expressed as
\[
\int_{LT}^{\infty} \phi_{\nu\nu}(\tau)e^{-2\pi f_\tau} d\tau = \sum_{m=L}^{\infty} \int_{mT}^{(m+1)T} \phi_{\nu\nu}(\tau)e^{-j2\pi f_\tau} d\tau
\]

Now, let \( \tau = \xi + mT \).

\[
\int_{LT}^{\infty} \phi_{\nu\nu}(\tau)e^{-2\pi f_\tau} d\tau = \sum_{m=L}^{\infty} \int_{0}^{T} \phi_{\nu\nu}(\xi + mT)e^{-j2\pi f(\xi + mT)} d\xi
\]
\[
= \sum_{m=L}^{\infty} \int_{0}^{T} \lambda(\xi)[\psi(jh)]^{m-L} e^{-j2\pi f(\xi + mT)} d\xi
\]

let \( n = m - L \)  
\[
= \sum_{n=0}^{\infty} \psi^n(jh)e^{-j2\pi f_{nT}} \int_{0}^{T} \lambda(\xi)e^{-j2\pi f(\xi + LT)} d\xi \quad \text{(A)}
\]
4.4.2 Power Spectra of CPFSK and CPM Signals

A property of the characteristic function is $|\psi(jh)| \leq 1$ since:

$$\psi(jh) = E(e^{j\pi h l_n}) = \sum_{n=-(M-1)}^{M-1} P_n e^{j\pi h n}$$

For values of $h$ for which $|\psi(jh)| < 1$, the summation in equation converges and yields

$$\sum_{n=0}^{\infty} \psi^n(jh)e^{-j2\pi fnT} = \frac{1}{1 - \psi(jh)e^{-j2\pi fT}}$$

In this case, equation (A) reduces to

$$\int_{LT}^{\infty} \phi_{\nu\nu}(\tau)e^{-j2\pi f_n \tau} d\tau = \frac{1}{1 - \psi(jh)e^{-j2\pi fT}} \int_{0}^{T} \phi_{\nu\nu}(\xi + LT)e^{-j2\pi f(\xi + LT)} d\xi$$

Note that: $\phi_{\nu\nu}(\xi + mT) = [\psi(jh)]^{m-L} \lambda(\xi)$, $m \geq L$, $0 \leq \xi \leq T$

For $m = L$, $\phi_{\nu\nu}(\xi + LT) = \lambda(\xi)$
By combining equations, we obtain the power density spectrum of the CPM signal in the form

\[
\Phi_{vv}(f) = 2 \Re \left[ \int_0^{LT} \overline{\phi_{vv}(\tau)} e^{-j2\pi f \tau} d\tau + \frac{1}{1 - \psi(jh)} e^{-j2\pi fT} \int_{LT}^{(L+1)T} \overline{\phi_{vv}(\tau)} e^{-j2\pi f \tau} d\tau \right]
\]

This is the desired result when \(|\psi(jh)| < 1\).

For values of \(h\) for which \(|\psi(jh)| = 1\), e.g., \(h=K\), where \(K\) is an integer, we can set

\[
\psi(jh) = e^{j2\pi \nu}, \quad 0 \leq \nu < 1
\]

then (summation in Equation A)

\[
\sum_{n=0}^{\infty} e^{-j2\pi T(f-\nu/T)n} = \frac{1}{2} + \frac{1}{2T} \sum_{n=-\infty}^{\infty} \delta(f - \frac{\nu}{T} - \frac{n}{T}) - j \frac{1}{2} \cot \pi T(f - \frac{\nu}{T})
\]
Thus, the power density spectrum now contains impulses located at frequencies

\[ f_n = \frac{n + v}{T}, \quad 0 \leq v \leq 1, \quad n = 0, 1, 2, \ldots \]

The result can be combined with equations (A) & (B) to obtain the entire power density spectrum, which includes both a continuous spectrum component and a discrete spectrum component.

Let us return to the case for \(|\psi(jh)| < 1\). When symbols are equally probable, i.e.,

\[ P_n = \frac{1}{M} \quad \text{for all } n \]
4.4.2 Power Spectra of CPFSK and CPM Signals

Characteristic function simplifies to the form

\[
\psi(jh) = \frac{1}{M} \sum_{n=-(M-1)}^{M-1} e^{jn\pi h} = \frac{1}{M} \sin M\pi h
\]

Note that in this case \(\psi(jh)\) is real.

The autocorrelation function also simplifies in this case to

\[
\overline{\phi_{\nu\nu}}(\tau) = \frac{1}{2T} \int_{0}^{T} \prod_{k=1-L}^{[\tau/T]} \frac{\sin 2\pi hM [q(t+\tau-kT) - q(t-kT)]}{\sin 2\pi h [q(t+\tau-kT) - q(t-kT)]} dt
\]
4.4.2 Power Spectra of CPFSK and CPM Signals

The power density spectrum reduces to

\[
\Phi_{\nu \nu}(f) = 2 \left[ \int_0^{LT} \phi_{\nu \nu}(\tau) \cos 2\pi f \tau d\tau \right.
\]

\[
+ \frac{1 - \psi(jh) \cos 2\pi fT}{1 + \psi^2(jh) - 2\psi(jh) \cos 2\pi fT} \int_{LT}^{(L+1)T} \phi_{\nu \nu}(\tau) \cos 2\pi f \tau d\tau
\]

\[
- \frac{\psi(jh) \sin 2\pi fT}{1 + \psi^2(jh) - 2\psi(jh) \cos 2\pi fT} \int_{LT}^{(L+1)T} \phi_{\nu \nu}(\tau) \sin 2\pi f \tau d\tau \left. \right]
\]

Power density spectrum of CPFSK.

A closed-form expression for the power density spectrum can be obtained when the pulse shape \( g(t) \) is rectangular and zero outside the interval \([0, T]\). In this case, \( q(t) \) is linear for \( 0 \leq t \leq T \).
4.4.2 Power Spectra of CPFSK and CPM Signals

Power density spectrum of CPFSK (cont.)

The resulting power spectrum

\[
\Phi_{vv}(f) = T \left[ \frac{1}{M} \sum_{n=1}^{M} A_n^2(f) + \frac{2}{M^2} \sum_{n=1}^{M} \sum_{m=1}^{M} B_{nm}(f) A_n(f) A_m(f) \right]
\]

where

\[
A_n(f) = \frac{\sin \pi \left[ fT - \frac{1}{2} (2n - 1 - M)h \right]}{\pi \left[ fT - \frac{1}{2} (2n - 1 - M)h \right]}
\]

\[
B_{nm}(f) = \frac{\cos(2\pi fT - \alpha_{nm}) - \psi \cos \alpha_{nm}}{1 + \psi^2 - 2\psi \cos 2\pi fT}
\]

\[
\alpha_{nm} = \pi h(m + n - 1 - M)
\]

\[
\psi \equiv \psi(jh) = \frac{\sin M\pi h}{M \sin \pi h}
\]
4.4.2 Power Spectra of CPFSK and CPM Signals

Power density spectrum of binary CPFSK

(a) Spectral density for two-level CPFSK

(b) Spectral density for two-level CPFSK

(c) Spectral density for two-level CPFSK

(d) Spectral density for two-level CPFSK
4.4.2 Power Spectra of CPFSK and CPM Signals

- **Power density spectrum of quaternary CPFSK**

  ![Spectral density for four-level CPFSK](image1)

  ![Spectral density for four-level CPFSK](image2)

  ![Spectral density for four-level CPFSK](image3)

- Note that only one-half of the bandwidth occupancy is shown in these graphs.

- The graphs illustrate that the spectrum of CPFSK is relatively smooth and well confined for $h<1$. 
4.4.2 Power Spectra of CPFSK and CPM Signals

- **Power density spectrum of octal CPFSK**

  As $h$ approaches unity, the spectra become very peaked and, for $h=1$ when $|\psi|=1$, we find that impulses occur at $M$ frequencies. When $h>1$, the spectrum becomes much broader.

- In communication systems where CPFSK is used, the modulation index is designed to conserve bandwidth, so that $h<1$. 
4.4.2 Power Spectra of CPFSK and CPM Signals

Power density spectrum of CPFSK (cont.)

- The special case of binary CPFSK with $h=1/2$ (of $f_d = 1/4T$) and $\psi = 0$ corresponds to MSK. In this case, the spectrum of the signal is

$$\Phi_{\nu \nu}(f) = \frac{16A^2T}{\pi^2} \left( \frac{\cos 2\pi f T}{1 - 16f^2 T^2} \right)^2$$

where the signal amplitude $A=1$.

- In contrast, the spectrum of four-phase offset QPSK (OQPSK) with a rectangular pulses $g(t)$ of duration $T$ is

$$\Phi_{\nu \nu}(f) = A^2T \left( \frac{\sin \pi f T}{\pi f T} \right)^2$$
Power density spectrum of CPFSK (cont.)

- If we compare these spectral characteristics, we should normalize the frequency variable by the bit rate or the bit interval $T_b$.
- Since MSK is binary FSK, it follows that $T = T_b$. On the other hand, in OQPSK, $T = 2T_b$ so that

$$
\Phi_{uv}(f) = 2A^2T_b \left( \frac{\sin 2\pi f T_b}{2\pi f T_b} \right)^2
$$
4.4.2 Power Spectra of CPFSK and CPM Signals

Power density spectrum of CPFSK (cont.)

Note that the main lobe of MSK is 50 percent wider than that for OQPSK. The side lobes in MSK fall off considerably faster.

If we compare the bandwidth $W$ that contains 99 percent of the total power, we find that $W = 1.2T_b$ for MSK and $W \approx 8/T_b$ for OQPSK.
4.4.2 Power Spectra of CPFSK and CPM Signals

- Power density spectrum of CPFSK (cont.)
  - MSK has a narrower spectral occupancy when viewed in terms of fractional out-of-band power above $fT_b = 1$.
  - Graphs for the fractional out-of-band power for OQPSK and MSK are shown.
  - Note that MSK is significantly more bandwidth-efficient than QPSK.
4.4.2 Power Spectra of CPFSK and CPM Signals

- Power density spectrum of CPFSK (cont.)
  - Even greater bandwidth efficiency than MSK can be achieved by reducing the modulation index. However, the FSK signals will no longer be orthogonal and there will be an increase in the error probability.

- Spectral characteristics of CPM.
  - The bandwidth occupancy of CPM depends on the choice of the modulation index $h$, the pulse shape $g(t)$, and the number of signals $M$.
  - Small values of $h$ result in CPM signals with relatively small bandwidth occupancy, while large values of $h$ result in signals with large bandwidth occupancy.
Spectral characteristics of CPM. (cont.)

The use of smooth pulses such as raised cosine pulses of the form

\[
g(t) = \begin{cases} 
  \frac{1}{2LT} \left(1 - \cos \frac{2\pi t}{LT}\right) & (0 \leq t \leq LT) \\
  0 & \text{(otherwise)}
\end{cases}
\]

$L = 1$ for full response and $L > 1$ for partial response, result in smaller bandwidth occupancy and, hence, greater bandwidth efficiency than the use of rectangular pulses.
Spectral characteristics of CPM. (cont.)

For example, the following figure illustrates the power density spectrum for binary CPM with different partial response raised cosine (LRC) pulses when $h=\frac{1}{2}$. (For comparison, the spectrum of binary CPFSK is also shown.)

Note that as $L$ increases the pulse, $g(t)$ becomes smoother and the corresponding spectral occupancy of the signal is reduced.
4.4.2 Power Spectra of CPFSK and CPM Signals

Spectral characteristics of CPM. (cont.)

- The effect of varying the modulation index in a CPM signal is illustrated in Figure for the case of $M=4$.

- Note that these spectral characteristics are similar to the ones illustrated previously for CPFSK, except that these spectra are narrower due to the use of a smoother pulse shape.
4.4.2 Power Spectra of CPFSK and CPM Signals

- Spectral characteristics of CPM. (cont.)
  - Finally, we illustrate the fractional out-of-band power for two-amplitude CPFSK with several different values of $h$. 

![Graph showing fractional out-of-band power vs. normalized frequency $fT$]
In this section, we consider the spectral characteristics of linearly modulated signals that have memory that can be modeled by a Markov chain.

For signals that are generated by a Markov chain with transition probability matrix $P$, the power density spectrum of the modulated signal may be expressed in the general form

$$
\Phi(f) = \frac{1}{T^2} \sum_{n=-\infty}^{\infty} \left| \sum_{i=1}^{K} p_i S_i \left( \frac{n}{T} \right) \right|^2 \delta\left( f - \frac{n}{T} \right) + \frac{1}{T} \sum_{i=1}^{K} p_i |S_i'(f)|^2
$$

$$
+ \frac{2}{T} \text{Re} \left[ \sum_{i=1}^{K} \sum_{j=1}^{K} p_i S_j''(f) S_j'(f) P_{ij}(f) \right]
$$
4.4.3 Power Spectra of Modulated Signals with Memory

- $S_i(f)$ is the Fourier transform of the signal waveform $s_i(t)$

$$s_i'(t) = s_i(t) - \sum_{k=1}^{K} p_k s_k(t)$$

- $P_{ij}(f')$ is the Fourier transform of the discrete-time sequence $p_{ij}(n)$,

$$P_{ij}(f) = \sum_{n=1}^{\infty} p_{ij}(n)e^{-j2\pi nfT}$$

and $K$ is the number of states of the modulator.

- $p_{ij}(n)$ denotes the probability that the signal $s_j(t)$ is transmitted $n$ signaling intervals after the transmission of $s_i(t)$. Hence, $\{p_{ij}(n)\}$ are the transition probabilities in the transition probability matrix $P^n$. Note that $p_{ij}(1) = p_{ij}$. 


When there is no memory in the modulation method, the signal waveform transmitted on each signaling interval is independent of the waveforms transmitted in previous signaling intervals.

The transition probability matrix is replaced by

\[
P = \begin{bmatrix}
p_1 & p_2 & \cdots & p_K \\
p_1 & p_2 & \cdots & p_K \\
\vdots & \vdots & \ddots & \vdots \\
p_1 & p_2 & \cdots & p_K \\
\end{bmatrix}
\]

and we impose the condition that \( P^n = P \) for all \( n \geq 1 \).
4.4.3 Power Spectra of Modulated Signals with Memory

Under these conditions, the expression for the power density spectrum becomes a function of the stationary state probabilities \( \{p_i\} \) only,

\[
\Phi(f) = \frac{1}{T^2} \sum_{n=-\infty}^{\infty} \left| \sum_{i=1}^{K} p_i S_i \left( \frac{n}{T} \right) \right|^2 \delta(f - n/T)
\]

\[
+ \frac{1}{T} \sum_{i=1}^{K} p_i (1 - p_i) |S_i(f)|^2
\]

\[
- \frac{2}{T} \sum_{i=1}^{K} \sum_{j=1 \atop i<j}^{K} p_i p_j \text{Re}[S_i(f)S_j^*(f)]
\]
We also make the observation that the first term in the expression for the power density spectrum. This line spectrum vanishes when

\[ \sum_{i=1}^{K} p_i S_i \left( \frac{n}{T} \right) = 0 \]

The condition is usually imposed in the design of practical digital communication systems and is easily satisfied by an appropriate choice of signaling waveforms.

Now, let us determine the power density spectrum of the baseband-modulated signals described in Section 4.3.2.

First, the NRZ signal is characterized by the two waveforms \( s_1(t) = g(t) \) and \( s_2(t) = -g(t) \), where \( g(t) \) is a rectangular pulse of amplitude \( A \).
For $K = 2$, 

$$\Phi(f) = \frac{(2p - 1)^2}{T^2} \sum_{n=-\infty}^{\infty} \left| G\left(\frac{n}{T}\right) \right|^2 \delta(f - \frac{n}{T}) + \frac{4p(1-p)}{T} \left| G(f) \right|^2$$

where 

$$\left| G(f) \right|^2 = (AT)^2 \left( \frac{\sin \pi f T}{\pi f T} \right)^2$$

Observe that when $p = \frac{1}{2}$, the line spectrum vanishes and $\Phi(f)$ reduces to 

$$\Phi(f) = \frac{1}{T} \left| G(f) \right|^2$$
The NRZI signal is characterized by the transition probability matrix

\[
P = \begin{bmatrix}
1 & 1 \\
2 & 2 \\
1 & 1 \\
2 & 2
\end{bmatrix}
\]

notice that in this case \( P_n = P \) for all \( n \geq 1 \).

Consequently, the power density spectrum for the NRZI signal is identical to the spectrum of the NRZ signal.
Delay modulation has a transition probability matrix

\[
P = \begin{bmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0
\end{bmatrix}
\]

and stationary state probabilities \( p_i = \frac{1}{4} \) for \( i = 1, 2, 3, 4 \).

Powers of \( P \) may be obtained by use of the relation

\[
P^4 \rho = -\frac{1}{4} \rho
\]

\( \rho \) is the signal correlation matrix with elements

\[
\rho_{ij} = \frac{1}{T} \int_0^T s_i(t)s_j(t)dt
\]
The four signals \( \{s_i(t), i = 1, 2, 3, 4\} \) are shown. It is easily seen that

\[
\rho = \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix}
\]

Consequently, powers of \( P \) can be generated from the relation

\[
P^{k+4} \rho = -\frac{1}{4} P^k \rho , \quad k > 1
\]
4.4.3 Power Spectra of Modulated Signals with Memory

- The power density spectrum of delay modulation may be expressed in the form

\[ \Phi(f) = \frac{1}{2\psi^2 (17 + 8 \cos 8\psi)} (23 - 2 \cos \psi - 22 \cos 2\psi - 12 \cos 3\psi + 5 \cos 4\psi + 12 \cos 5\psi + 2 \cos 6\psi - 8 \cos 7\psi + 2 \cos 8\psi) \]

where \( \psi = \pi fT \).

- The spectra of these baseband signals are illustrated in Figure.
4.4.3 Power Spectra of Modulated Signals with Memory

- Observe that the spectra of the NRZ and NRZI signals peak at \( f = 0 \).
- Delay modulation has a narrower spectrum and a relatively small zero-frequency content. Its bandwidth occupancy is significantly smaller than that of the NRZ signal.
- These two characteristics make delay modulation an attractive choice for channels that do not pass DC, such as magnetic recording media.