Problem 10.3

Entropy of the source is

\[
H(S) = p_0 \log_2\left(\frac{1}{p_0}\right) + p_1 \log_2\left(\frac{1}{p_1}\right) + p_2 \log_2\left(\frac{1}{p_2}\right) + p_3 \log_2\left(\frac{1}{p_3}\right)
\]

\[
= \frac{1}{3} \log_2(3) + \frac{1}{6} \log_2(6) + \frac{1}{4} \log_2(4) + \frac{1}{4} \log_2(4)
\]

\[
= 0.528 + 0.431 + 0.5 + 0.5
\]

\[
= 1.959 \text{ bits}
\]

Problem 10.4

Let \( X \) denote the number showing on a single roll of a dice. With a dice having six faces, we note that \( p_X \) is 1/6. Hence, the entropy of \( X \) is

\[
H(X) = p_X \log_2\left(\frac{1}{p_X}\right)
\]

\[
= \frac{1}{6} \log_2(6) = 0.431 \text{ bits}
\]
Problem 10.5

The entropy of the quantizer output is

\[ H = - \sum_{k=1}^{4} P(X_k) \log_2 P(X_k) \]

where \( X_k \) denotes a representation level of the quantizer. Since the quantizer input is Gaussian with zero mean, and a Gaussian density is symmetric about its mean, we find that

\[ P(X_1) = P(X_4) \]
\[ P(X_2) = P(X_3) \]

The representation level \( X_1 = 1.5 \) corresponds to a quantizer input \( +1 \leq Y < \infty \). Hence,

\[ P(X_1) = \int_{1}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \]
\[ = \frac{1}{2} - \frac{1}{2} \text{erf}\left(\frac{4}{\sqrt{2}}\right) \]
\[ = 0.1611 \]

The representation level \( X_2 = 0.5 \) corresponds to the quantizer input \( 0 \leq Y < 1 \). Hence,

\[ P(X_2) = \int_{0}^{1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \]
\[ = \frac{1}{2} \text{erf}\left(\frac{4}{\sqrt{2}}\right) \]
\[ = 0.3389 \]

Accordingly, the entropy of the quantizer output is

\[ H = -2 \left[ 0.1611 \log_2 \left( \frac{1}{0.1611} \right) + 0.3389 \log_2 (0.3389) \right] \]
\[ = 1.91 \text{ bits} \]
Problem 10.6

(a) For a discrete memoryless source:
\[ P(\sigma_i) = P(s_{i_1})P(s_{i_2})...P(s_{i_n}) \]

Noting that \( M = K^n \), we may therefore write
\[ \sum_{i=0}^{M-1} P(\sigma_i) = \sum_{i_1=0}^{K-1} \sum_{i_2=0}^{K-1} ... \sum_{i_n=0}^{K-1} P(s_{i_1})P(s_{i_2})...P(s_{i_n}) \]
\[ = \sum_{i_1=0}^{K-1} \sum_{i_2=0}^{K-1} ... \sum_{i_n=0}^{K-1} P(s_{i_1})P(s_{i_2})...P(s_{i_n}) \]
\[ = 1 \]

(b) For \( k = 1, 2, ..., n \), we have
\[ \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \left( \frac{1}{P_{i_k}} \right) = \sum_{i=0}^{M-1} P(s_{i_1})P(s_{i_2})...P(s_{i_n}) \log_2 \left( \frac{1}{P_{i_k}} \right) \]

For \( k = 1 \), say, we may thus write
\[ \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \left( \frac{1}{P_{i_1}} \right) = \sum_{i_2=0}^{K-1} \sum_{i_3=0}^{K-1} ... \sum_{i_n=0}^{K-1} P(s_{i_1})P(s_{i_2})...P(s_{i_n}) \log_2 \left( \frac{1}{P_{i_1}} \right) \]
\[ = \sum_{i=0}^{K-1} P(s_{i_1}) \log_2 \left( \frac{1}{P_{i_1}} \right) \]
\[ = H(S) \]

Clearly, this result holds not only for \( k = 1 \), but also for \( k = 2, ..., n \).

(c) \[ H(S^n) = \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(\sigma_i)} \]
\[ = \sum_{i=0}^{M-1} P(\sigma_i) \log_2 P(s_{i_1})P(s_{i_2})...P(s_{i_n}) \]
\[ = \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \left( \frac{1}{P(s_{i_1})P(s_{i_2})...P(s_{i_n})} \right) \]
\[ = \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \left( \frac{1}{P(s_{i_1})} + P(\sigma_i) \log_2 \frac{1}{P(s_{i_2})} \right) \]
\[ + ... + \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \left( \frac{1}{P(s_{i_n})} \right) \]

Using the result of part (b), we thus get
\[ H(S^n) = H(S) + H(S) + ... + H(S) \]
\[ = nH(S) \]
Problem 10.7

a)

A prefix code is defined as a code in which no code word is the prefix of any other code word. By inspection, we see therefore that codes I and IV are prefix codes, whereas codes II and III are not.

To draw the decision tree for a prefix code, we simply begin from some starting node, and extend branches forward until each symbol of the code is represented. We thus have:

b) To be done
Problem 10.9

The Huffman code is therefore

\begin{align*}
    s_0 & \quad 10 \\
    s_1 & \quad 11 \\
    s_2 & \quad 001 \\
    s_3 & \quad 010 \\
    s_4 & \quad 011 \\
    s_5 & \quad 0000 \\
    s_6 & \quad 0001
\end{align*}

The average code-word length is

\[
    L = \sum_{k=0}^{6} p_k l_k
\]

\[
    = 0.25(2)(2) + 0.125(3)(3) + 0.0625(4)(2)
\]

\[
    = 2.625
\]

The entropy of the source is

\[
    H(S) = \sum_{k=0}^{6} p_k \log_2 \left( \frac{1}{p_k} \right)
\]

\[
    = 0.25(2) \log_2 \left( \frac{1}{0.25} \right) + 0.125(3) \log_2 \left( \frac{1}{0.125} \right)
\]

\[
    + 0.0625(2) \log_2 \left( \frac{1}{0.0625} \right)
\]

\[
    = 2.625
\]

The efficiency of the code is therefore
\[ \eta = \frac{H(S)}{L} = \frac{2.625}{2.625} = 1 \]

We could have shown that the efficiency of the code is 100\% by inspection since

\[ \eta = \frac{\sum_{k=0}^{\infty} p_k \log_2(1/p_k)}{\sum_{k=0}^{\infty} p_k l_k} \]

where \( l_k = \log_2(1/p_k) \).

**Problem 10.13**

\[ p(x_0) = p(x_1) = \frac{1}{2} \]

\[ p(y_0) = (1-p)p(x_0) + p \ p(x_1) \]
\[ = (1-p)\left(\frac{1}{2}\right) + p\left(\frac{1}{2}\right) \]
\[ = \frac{1}{2} \]

\[ p(y_1) = p \ p(x_0) + (1-p)p(x_1) \]
\[ = p\left(\frac{1}{2}\right) + (1-p)\left(\frac{1}{2}\right) \]
\[ = \frac{1}{2} \]
Problem 10.20

The generator matrix for the (7,4) Hamming code is

\[
G = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

The parity-check matrix is

\[
H = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

Hence,

\[
HG^T = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad \text{mod}-2
\]
Problem 10.21

(a) Viewing the matrix

\[ H = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & \vdots & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 1 & 1 & 1 \end{bmatrix} \]

as a generator matrix, we may define the code vector \( c \) in terms of the message vector \( m \) as

\[ c = m H \]

The message word length is

\[ n - k = 7 - 4 = 3 \]

Hence, we may construct the following table

<table>
<thead>
<tr>
<th>Message word</th>
<th>Code word</th>
<th>Hamming weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0</td>
<td>0 0 0 0 0 0</td>
<td>0</td>
</tr>
<tr>
<td>0 0 1</td>
<td>0 0 1 0 1 1</td>
<td>4</td>
</tr>
<tr>
<td>0 1 0</td>
<td>0 1 0 1 1 0</td>
<td>4</td>
</tr>
<tr>
<td>0 1 1</td>
<td>0 1 1 0 0 1</td>
<td>4</td>
</tr>
<tr>
<td>1 0 0</td>
<td>1 0 0 1 0 1</td>
<td>4</td>
</tr>
<tr>
<td>1 0 1</td>
<td>1 0 1 1 0 0</td>
<td>4</td>
</tr>
<tr>
<td>1 1 0</td>
<td>1 1 0 0 1 0</td>
<td>4</td>
</tr>
<tr>
<td>1 1 1</td>
<td>1 1 1 0 0 1</td>
<td>5</td>
</tr>
</tbody>
</table>

(b) The minimum value of the Hamming weight defines the Hamming distance of the dual code as

\[ d_{\text{min}} = 4 \]
Problem 10.25

The generator polynomials are

\[ g^{(1)}(X) = 1 + X + X^2 + X^3 \]
\[ g^{(2)}(X) = 1 + X + X^3 \]

The message polynomial is

\[ m(X) = 1 + X^2 + X^3 + X^4 + \ldots \]

Hence,

\[ c^{(1)}(X) = g^{(1)}(X)m(X) = 1 + X + X^3 + X^4 + X^5 + \ldots \]

\[ c^{(2)}(X) = g^{(2)}(X)m(X) = 1 + X + X^2 + X^3 + X^6 + X^7 + \ldots \]

Hence,

\[ \{c^{(1)}\} = 1, 1, 0, 1, 1, 1, \ldots \]
\[ \{c^{(2)}\} = 1, 1, 1, 1, 0, 0, \ldots \]

The encoder output is therefore 11, 11, 01, 11, 10, 10.