ELEC 546
Lecture #9
Orthogonal Frequency Division Multiplexing (OFDM): Basic OFDM System
Outline

- Motivations
- Diagonalization of Vector Channels
- Transmission of one OFDM Symbol
- Transmission of sequence of OFDM Symbols
- Sample and Symbol Time Synchronization
- Carrier Frequency Synchronization
- Peak-to-Average Power Ratio (PAPR) Issue
OFDM: Motivations

• Treat a Wideband FS fading channel as Multiple Narrowband Flat fading channels
• Change in TX so that RX does not suffer from ISI
• Use FEC with codeword span across all sub-channels achieve Frequency Diversity, but with no ISI problem
OFDM: Motivations

- Motivation:
  - Split a frequency selective fading channel into multiple, say N=1024, narrowband flat fading sub-channels
  - Send the bits over these sub-channels in parallel
OFDM: Motivation

• Problems:
  – Multiple transmitter front ends (mixer, modulator, etc)
  – require guard bands

• Solutions:
  – Do all these in digital domain using a wide baseband signal
  – Use DFT (discrete Fourier transform) to create the baseband equivalent of the transmit signal and then up-convert it to the center frequency using one front end
  – As DFT is an orthogonal transformation, no guard band is needed
Diagonalization of Vector Channels

• Consider a Vector Channel with input $x$ and output $y$

$$ y = Hx + n $$

$$ y_i = \sum_{j=1}^{N} H_{i,j} x_j + n_i $$

• Want to diagonalize it such that

$$ z = Uy = UHV_s + Un $$

$$ = D_H s + \tilde{n} $$

$$ z_i = d_i s_i + \tilde{n}_i \quad i = 1, \ldots, N $$
Maintaining the same SNR during Diagonalization

- U has to be Unitary to prevent noise enhancement

\[ E\tilde{n}^H = E U n^H U^H \]
\[ = N_0 U U^H \]
\[ = N_0 \quad \text{if } U \text{ is unitary} \]

- V has to be Unitary to maintain the same transmit power

\[ E x^H x = E s^H V^H V s \]
\[ = E s^H s \quad \text{if } V \text{ is unitary} \]
Advantage and Issues with Diagonalization

• Need to find $U$ and $V$ s.t. $UHV = D_H$ is diagonal
  – Decompose the vector channels into parallel channels with different gain (allow adaptive modulation, and TX optimization to be discussed in 2nd part of OFDM notes)

• $V$ depends on $H$ (TX needs to know the vector channel)
Diagonalization of ISI channel

- For frequency selective fading channel
  (\# of resolvable paths = 2), time
domain response has ISI \( \Rightarrow H \) is
  Toeplitz

\[
y_1 = h_0 x_1 + n_1, \\
y_2 = h_0 x_2 + h_1 x_1 + n_2, \\
y_3 = h_0 x_3 + h_1 x_2 + n_3, \ldots
\]

- \( \tilde{H} \) is a circulant matrix if

\[
\tilde{H}_{i,j} = \tilde{H}_{(i-j)N} \quad \tilde{H} = \begin{bmatrix}
h_0 & 0 & \cdots & \cdots & 0 \\
h_1 & h_0 & 0 & \cdots & 0 \\
0 & h_1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & h_1 & h_0 \\
\end{bmatrix}
\]
Diagonization of Circulant Matrix

• If $H$ is circulant, then

$$W^H \tilde{H} W = D_H; \quad W_{mn} = \frac{1}{\sqrt{N}} \exp\left(j\pi \frac{mn}{N}\right)$$

$$\left(D_H\right)_{nn} = \sum_{m=0}^{N-1} h_m e^{-j\pi \frac{mn}{N}}$$

• Note that THE $W$ that diagonalized $\tilde{H}$ is independent of $H$!
  – Hence, TX does not need to know $\tilde{H}$!!

• Use cyclic prefix to create an effective circulant matrix
Cyclic Prefix

• Instead of transmitting $x=\mathbf{W}s$, transmit $\tilde{x} = [x_N \quad x]^T$
• Then,

$$H\tilde{x} = \begin{bmatrix} h_1 & h_0 & 0 & \cdots & \cdots & 0 \\ 0 & h_1 & h_0 & \cdots & \cdots & \vdots \\ \vdots & \ddots & h_1 & h_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & h_1 \\ 0 & \cdots & \cdots & 0 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} x_N \\ x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 0 & h_0 & 0 & \cdots & \cdots & 0 & h_1 \\ 0 & h_1 & h_0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \ddots & h_1 & h_0 & \cdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & h_1 & h_0 \\ 0 & \cdots & \cdots & 0 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \tilde{H}x$$

• Transmit a length $N+N_s$ vector for a length-$N$ data vector. Efficiency = $N/(N+N_s)$ with $N_s>L$ (ISI length)
• For $N=2^{10}=1024$, $N_s=10$, Efficiency $\sim 99\%$.
OFDM Transmission

\[ X = W_{IFFT} S \]

Serial to Parallel

\[ X \sim (N + N_s) \text{ samples} \]

Add Cyclic Prefix & Pulse Shaping

Mixer

Frequency Selective Channel

\[ \tilde{X} \sim (N + N_s) \text{ samples} \]

\[ \tilde{Y} \sim (N + N_s) \text{ samples} \]

\[ Z = W_{FFT} Y \sim N \text{ samples} \]
OFDM Transmission

- Time-domain modulation
  - Modulation Pulse overlaps in time
  - At ideal sampling position, there is no ISI
  - With timing offset $\rightarrow$ ISI
- OFDM
  - Modulation pulse overlaps in freq
  - At ideal sampling position (in freq), there is no ICI
  - With frequency offset $\rightarrow$ ICI
Transmission of a sequence of OFDM symbols

• Using a block of $N$ samples to create an OFDM symbol ($x=\text{Ws}$) and the cyclic prefix, ISI between samples within an OFDM symbol is eliminated.

• What happens to the intersymbol interference between OFDM symbols?
Cyclic Prefix

- Insert a Cyclic Prefix before every OFDM symbol
  - Cycle Prefix length $> \tau_{\text{max}}$
  - Overhead is $\tau_{\text{max}}/(NT_s)$ where $T_s=1/B$ is the sampling period, $B$ is the bandwidth and $N$ is the number of sub-carriers or points in the IFFT
    - the larger the $N$, the smaller the overhead!!
- If $\tau_{\text{max}}/T_s=N_s$. Then, there will have $N+N_s$ sampled points for every OFDM symbol
Cyclic Prefix

• If we just take the last $N$ points out of the $N+N_s$ points to do the FFT at the receiver,

• Then \[ y_n = x_n \otimes h_n \]

where $\otimes$ denotes circular convolution

and \( Y_k = H_k X_k \)

where $Y_k$, $X_k$, and $H_k$ are the DFT of $y_n$, $x_n$, and $h_n$, resp
Equivalent Channel of OFDM

- Using IFFT, FFT and cyclic prefix, the OFDM transforms a frequency selective fading channel (in time domain) to parallel channels (in frequency domain).

- Discussion:
  - Is OFDM optimal in capacity sense (capacity of frequency selective fading channels)?
Advantages of OFDM

• With cyclic prefix, we can eliminate ISI completely
• Provide frequency diversity
  – Forward error correcting code such as convolutional code with interleaver is needed as some sub-carriers will be in deep fade
• Potential
  – If the transmitter knows the channel conditions
    • can select only the good sub-carriers to transmit or transmit different numbers of bits based on the sub-carriers’ gains → Power water-filling in frequency domain.
• If the transmitter knows the channel, OFDM with bit allocation is better than the best equalizer (e.g. MLSE)
Effect of Timing Offsets

- Sample and Symbol Synchronization
  - Sampling time synchronization
  - Sampling Frequency needs to be correct, but sampling instance offset only leads to linear phase shift in the sub-channels’ gains. (which will be handled by channel estimation) $\rightarrow$ only 1X ADC is needed at the receiver
  - OFDM Symbol Synchronization
  - Determine the beginning of the OFDM symbol and the beginning of the cyclic prefix (to avoid Inter-OFDM symbol interference)
  - Use Cyclic Prefix

Compute Correlation between two intervals separated by $NT_s$
Effects of Frequency Offsets

• Carrier Frequency Synchronization
  – Carrier Frequency offset can cause significant inter-subcarrier interference
    (Similar to timing offset causing ISI in time domain modulation)
  – As there is no guard band, very small frequency offset can lead to large
    inter-subcarrier (or inter-subchannel) interference
  – Very important and performance is sensitive to this
Effects of Doppler Spread

- When $T_c >>$ OFDM symbol time, channel fading coefficients are quasi-static $\rightarrow$ slow fading scenario.

- When $T_c <$ OFDM symbol time, channel fading coefficients are no longer quasi-static.
  - For simplicity, consider flat fading channel.
  - Effect of fast fading is equivalent to the multiplication of OFDM signal by a time-domain window $h(t)$.
  
  $$y(t) = x(t)h(t) = F^{-1} \left[ \sum_{n=0}^{N_f-1} X_n \text{sinc}(f - n\Delta f) \right]$$

  - Equivalently, the effect of fast fading after FFT is the circular convolution of the $H(f)$ with $X(f)$. $\rightarrow$ the shape of the subcarrier is distorted from the SINC pulse (in freq domain) to a smeared subcarrier shape (as a result of the convolution). $\rightarrow$ Fast fading results in ICI.

$$Y(f) = X(f) * H(f) = \sum_{n=0}^{N_f-1} X_n \left( \text{sinc}(f - n\Delta f) * H(f) \right) = \sum_{n=0}^{N_f-1} X_n G(f - n\Delta f)$$
Peak-to-Average Power Ratio

• The data symbol, $s_i$, may be QPSK modulated (constant magnitude), but the transmitted samples, $x_i$, is the output of the IFFT and hence takes values over a wide range.

• Statistically, as $s_i$ are independent and has random phase, $x_i$ approaches an Gaussian distribution when $N$ is large

• Hence, high Peak-to-Average Power Ratio

$$PAPR = \frac{(\max_i x_i)^2}{E x_i^2}$$
Disadvantages of OFDM

• Overheads
  – Cyclic Prefix: can be reduced by increasing N
  – Power to transmit cyclic prefix: can be lower by increasing N

• Implementation issues
  – sensitivity to frequency offsets
    • especially when N is large and sub-carrier spacing is small
  – require highly linear power amp
    • high peak-to-average-power (PAP) ratio, especially when N is large
  – Poor Adjacent band rejection ~ 20/30dB only.
    • Q: In wireless LAN (802.11g), the AP can receive and decode packets on an adjacent channel. Why?

• Typical value for N is $2^7$ to $2^{11}$