Introduction to Orthogonal Frequency Division Multiplexing (OFDM)
Outline

- OFDM Overview
- OFDM System Model
- Orthogonality
- Multi-carrier Equivalent Implementation by Using Inverse Discrete Fourier Transform (IDFT)
- Cyclic Prefix (CP)
- Summary
OFDM Overview (2/3)

- **OFDM**
  - Orthogonal Frequency Division Multiplexing

The basic principle of OFDM is to split a high-rate data stream into a number of lower rate streams that are transmitted simultaneously over a number of sub-carriers.

It eliminates or alleviates the problem of multi-path channel fading effect, and low spectrum efficiency.
OFDM Overview (1/3)

- Single carrier (SC) vs. multi-carrier (MC)

  - Single carrier: data are transmitted over only one carrier
  - Selective fading
  - Multi-carrier: data are shared among several carriers and simultaneously transmitted
  - Flat fading per subcarrier
OFDM Overview (3/3)

- OFDM modulation

- Features
  - No intercarrier guard bands
  - Overlapping of bands
  - Spectral efficiency
  - Easy implementation by IDFTs
  - Very sensitive to synchronization
  - Problems of Peak to Average Power Ratio (PAPR)
OFDM Transmitter

\[ f_s = \frac{1}{\Delta t} \]

\[ f_n = f_0 + n\Delta f, \Delta f = \frac{1}{N\Delta t} \]

**Figure 1**
An OFDM system transmitter is shown in Figure 1.

The transmitted waveform $D(t)$ can be expressed as

$$D(t) = \sum_{n=0}^{N-1} \{a(n) \cos(2\pi f_n t) + b(n) \sin(2\pi f_n t)\}$$

(1)

where $f_n = f_0 + n\Delta f$ and $\Delta f = \frac{1}{N\Delta t}$

Using a two-dimensional digital modulation format, the data symbols $d(n)$ can be represented as $a(n) + jb(n)$

- $a(n)$: in-phase component
- $b(n)$: quadrature component
OFDM System Model (2/3)

\[ a(0) + 1 \]

\[ N \Delta t \]

\[ \Delta t \]

\[ f_s = \frac{1}{\Delta t} \]

\[ d(n) = \{a(n) + jb(n)\} \]

\[ a(0) \]

\[ b(0) \]

\[ \sin(2\pi f_0 t) \]

\[ \cos(2\pi f_0 t) \]

\[ \sin(2\pi f_{N-1} t) \]

\[ \cos(2\pi f_{N-1} t) \]

\[ a(N-1) \]

\[ b(N-1) \]

\[ D(t) \]

Channel
The serial data elements spaced by $\Delta t$ are grouped and used to modulate $N$ carriers. Thus they are frequency division multiplexed.

The signaling interval is then increased to $N\Delta t$, which makes the system less susceptible to channel delay spread impairments.

**Small-scale fading**
(Based on multipath time delay spread)

- **Flat Fading**
  1. BW of signal < BW of channel
  2. Delay spread < Symbol period

- **Frequency Selective Fading**
  1. BW of signal > BW of channel
  2. Delay spread > Symbol period
Consider a set of transmitted carriers as follows:

\[ \psi_n(t) = e^{j2\pi\left(f_0 + \frac{n}{N\Delta t}\right)t} \quad \text{for } n = 0, 1, \ldots, N - 1 \quad (2) \]

We now show that every two carriers are orthogonal to each other. <proof is in next page>

In summary:

\[ \int_a^b \psi_p(t)\psi_q^*(t)dt = \begin{cases} (b - a) & \text{for } p = q \\ 0 & \text{for } p \neq q \text{ and } (b - a) = N\Delta t \end{cases} \]
Orthogonality (2/3) --- Proof

\[ \int_{a}^{b} \psi_{p}(t)\psi_{q}^{*}(t)dt = \int_{a}^{b} e^{j2\pi(p-q)\frac{t}{N\Delta t}}dt \]

\[ = e^{j2\pi(p-q)\frac{b}{N\Delta t}} - e^{j2\pi(p-q)\frac{a}{N\Delta t}} \]

\[ = \frac{\left(j2\pi(p-q)/N\Delta t\right)}{j2\pi(p-q)/N\Delta t} \left(1 - e^{j2\pi(p-q)\frac{1}{N\Delta t}(a-b)}\right) \]

\[ = 0, \text{ for } p \neq q \text{ and } (b-a) = N\Delta t \]
Orthogonality (3/3)

Very Sensitive to Frequency Errors

Time domain

Frequency domain

Example of four subcarriers within one OFDM symbol

Spectra of individual subcarriers

$N\Delta t$
From above, we know that \( \{\psi_n(t)\} \) is the orthogonal signal set. An OFDM signal based on this orthogonal signal set can be written as:

\[
x(t) = \text{Re}\left\{ \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} d_{k,n} \psi_n(t - kT) \right\}
\]

(3)

where \( \psi_n(t) = e^{j2\pi f_n t} \) for \( n = 0, 1, 2, \ldots, N-1 \) \( 0 \leq t \leq T \)

\[
f_n = f_0 + \frac{n}{T}, \quad T = N\Delta t
\]

\[
d_{k,n} = a_{k,n} + jb_{k,n}
\]
Mathematical Expression of OFDM Signal (2/2)

- $T$: OFDM symbol duration
- $d_{k,n}$: transmitted data on the $n$-th carrier of the $k$-th symbol

$$x(t) = \text{Re}\left\{\sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} C_{k,n} \psi_n(t - kT)\right\}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} \left\{ a_{k,n} \cos\left(2\pi f_n(t - kT)\right) - b_{k,n} \sin\left(2\pi f_n(t - kT)\right) \right\}$$ \hspace{1cm} (4)

- If there is only one OFDM symbol (i.e. $k = 0$), it can be simplified as:

$$x(t) = \sum_{n=0}^{N-1} \left\{ a_n \cos\left(2\pi f_n t\right) - b_n \sin\left(2\pi f_n t\right) \right\}$$ \hspace{1cm} (5)
According to the structure of the transmitter, transmitting an OFDM signal requires $N$ oscillators, an extremely high cost for hardware implementation.

Solution: An equivalent method for transmitting OFDM signals is using inverse discrete Fourier transform (IDFT).
In general, each carrier can be expressed as:

\[ S_c(t) = A_c(t) e^{j(2\pi f_c t + \phi_c(t))} \]  \hspace{1cm} (6)

We assume that there are \( N \) carriers in the OFDM signal. Then the total complex signal \( S_s(t) \) can be represented by:

\[ S_s(t) = \frac{1}{N} \sum_{n=0}^{N-1} A_n(t) e^{j(2\pi f_n t + \phi_n(t))} \]  \hspace{1cm} (7)

where \( f_n = f_0 + n\Delta f \)

and \( A_n(t), \phi_n(t), f_n \) are amplitude, phase, carrier frequency of \( n \)-th carrier, respectively.
Then we sample the signal at a sampling frequency \(1/\Delta t\), and \(A_n(t)\) and \(\phi_n(t)\) becomes:

\[
\phi_n(t) = \phi_n \quad (8)
\]
\[
A_n(t) = A_n \quad (9)
\]

\[
S_s(k\Delta t) = \frac{1}{N} \sum_{n=0}^{N-1} A_n e^{j(2\pi(f_0+n\Delta f)k\Delta t+\phi_n)} \quad (10)
\]

Then the sampled signal can be expressed as:

\[
S_s(k\Delta t) = \frac{1}{N} \sum_{n=0}^{N-1} \left( A_n e^{j(2\pi f_0 k\Delta t+\phi_n)} \right) e^{j2\pi knk\Delta f\Delta t} \quad (11)
\]
The inverse discrete Fourier transform (IDFT) is defined as the following:

\[ f(k\Delta t) = \frac{1}{N} \sum_{n=0}^{N-1} F(n\Delta f) e^{j2\pi nk/N} \]  

Comparing eq.(11) and eq.(12), the condition must be satisfied in order to make eq.(11) an inverse Fourier transform relationship:

\[ \Delta f = \frac{1}{N\Delta t} \]
Multi-carrier Equivalent Implementation by using IDFT (5/6)

- If eq.(13) is satisfied,
  - \( A^e_{n}e^{j(2\pi f_{0}k\Delta t+\phi_{n})} \) is the frequency domain signal
  - \( S_{s}(k\Delta t) \) is the time domain signal
  - \( \Delta f \) is the sub-channel spacing
  - \( N\Delta t \) is the symbol duration in each sub-channel

- This outcome is the same as the result obtained in the system of Figure 1. Therefore IDFT can be used to generate an OFDM transmission signal.
Multi-carrier Equivalent Implementation by using IDFT (6/6)

\[ f_s = \frac{1}{\Delta t} \]

\[ f_n = f_0 + n \Delta f, \Delta f = \frac{1}{N \Delta t} \]

Input

\[ d(n) \]  

\( \{a(n) + jb(n)\} \)

S/P

\[ a(0) \]  

\[ b(0) \]

\[ \sin(2 \pi f_0 t) \]

\[ \cos(2 \pi f_0 t) \]

MUX

\[ D(t) \]  

\[ \text{Channel} \]

Input

\[ f_s = \frac{1}{\Delta t} \]

\[ f_n = f_0 + n \Delta f, \Delta f = \frac{1}{N \Delta f} \]

Output

\[ d(0) \]

\[ d(1) \]

\[ d(2) \]

\[ \ldots \]

\[ d(\ldots) \]

\[ \{a(n) + jb(n)\} \]

S/P

IFFT (IDFT)

P/S

\[ D(t) \]  

\[ \text{Channel} \]
In multipath channel, delayed replicas of previous OFDM signal lead to ISI between successive OFDM signals.

Solution: Insert a guard interval between successive OFDM signals.
Guard interval leads to intercarrier interference (ICI) in OFDM demodulation.

In DFT interval, difference between two subcarriers does not maintain integer number of cycles ➔ loss of orthogonality.

Delayed version of subcarrier 2 causes ICI in the process of demodulating subcarrier 1.
Cyclic Prefix

Cyclic prefix (CP) : A copy of the last part of OFDM signal is attached to the front of itself.
All delayed replicas of subcarriers always have an integer number of cycles within DFT interval \( \Rightarrow \) no ICI
OFDM Receiver

\[ \tilde{r}(n) \xrightarrow{\text{Remove CP}} r(n) \xrightarrow{\text{S/P}} \xrightarrow{\text{FFT (DFT)}} \xrightarrow{\text{P/S}} \text{Output data Symbols} \]
One of the most important reasons to do OFDM is the efficient way it deals with multipath delay spread.

To eliminate inter-symbol interference (ISI), a guard time or cyclic prefix is introduced for each OFDM symbol.

The length of the guard time or cyclic prefix is chosen to be larger than the delay spread.
Bandwidth Efficiency

In a classical parallel system, the channel is divided into \( N \) non-overlapping sub-channels to avoid inter-carrier interference (ICI).

The diagram for bandwidth efficiency of OFDM system is shown below:
Summary

The advantage of the DFT-based OFDM system:
- The use of IDFT/DFT can reduce the computation complexity.
- The orthogonality between the adjacent sub-carriers will make the use of transmission bandwidth more efficient.
- The guard interval or cyclic prefix is used to resist the inter-symbol interference (ISI).
- The main advantage of the OFDM transmission technique is its high performance even in frequency selective channels.

The drawbacks of the OFDM system:
- It is highly vulnerable to synchronization errors.
- Peak to Average Power Ratio (PAPR) problems.